

## A problem on track runners

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### Abstract

Consider the circle  $C$  of length 1 and a circular arc  $A$  of length  $\ell < 1$ . It is shown that there exists  $k = k(\ell) \in \mathbb{N}$ , and a schedule for  $k$  runners along the circle with  $k$  distinct but constant positive speeds so that at any time  $t \geq 0$ , at least one of the  $k$  runners is *not* in  $A$ .

**Keywords:** Kronecker’s theorem, rational independence, track runners, multi-agent patrolling, idle time.

### 1 Introduction

In the classic *lonely runner conjecture*, introduced by Wills [11] and Cusick [4],  $k$  agents run clockwise along a circle of length 1, starting from the same point at time  $t = 0$ . They have distinct but constant speeds. A runner is called *lonely* when he/she is at distance of at least  $\frac{1}{k}$  from all other runners (along the circle). The conjecture asserts that each runner  $a_i$  is lonely at some time  $t_i \in (0, \infty)$ . The conjecture has only been confirmed for up to  $k = 7$  runners [1, 2]. A recent survey [7] lists a few other related problems.

Recently, some problems with similar flavor have appeared in the context of *multi-agent patrolling*, in some one-dimensional scenarios [3, 5, 6, 9, 10]. Suppose that  $k$  mobile agents with (possibly distinct) maximum speeds  $v_i$  ( $i = 1, \dots, k$ ) are in charge of *patrolling* a closed or open fence (modeled by a circle or a line segment). The movement of the agents over the time interval  $[0, \infty)$  is described by a *patrolling schedule* (or *guarding schedule*), where the speed of the  $i$ th agent, ( $i = 1, \dots, k$ ), may vary between zero and its maximum value  $v_i$  in any of the two directions along the fence. Given a closed or open fence of length  $\ell$  and maximum speeds  $v_1, \dots, v_k > 0$  of  $k$  agents, the goal is to find a patrolling schedule that minimizes the *idle time*, defined as the longest time interval in  $[0, \infty)$  during which some point along the fence remains unvisited, taken over all points. Several basic problems are open, such as the following: It is *not* known how to efficiently decide, given  $v_1, \dots, v_k > 0$ , and  $\ell, \tau > 0$  whether  $k$  agents with these

maximum speeds can ensure an idle time at most  $\tau$  when patrolling a segment of length  $\ell$ .

This note is devoted to a question on track runners. As customary, we consider the unidirectional circular track; for convenience we assume runners run clockwise. In the spirit of the lonely runner conjecture, we posed the following question in [7]:

Assume that  $k$  runners  $1, 2, \dots, k$ , with distinct but constant speeds, run clockwise along a circle of length 1, starting from arbitrary points. Assume also that a certain half of the circular track (or any other fixed circular arc) is in the shade at all times. Does there exist a time when all runners are in the shade along the track?

Here we answer the question in the negative: the statement does not hold even if the shaded arc almost covers the entire track, e.g., has length 0.999, provided  $k$  is large enough.

**Notation and terminology.** We parameterize a circle of length  $\ell$  by the interval  $[0, \ell]$ , where the endpoints of the interval  $[0, \ell]$  are identified. A *unit* circle is a circle of unit length  $C = [0, 1] \bmod 1$ . A *schedule* of  $k$  agents consists of  $k$  functions  $f_i : [0, \infty) \rightarrow [0, \ell]$ , for  $i = 1, \dots, k$ , where  $f_i(t) \bmod \ell$  is the position of agent  $i$  at time  $t$ . Each function  $f_i$  is continuous, piecewise differentiable, and its derivative (speed) is bounded by  $|f'_i| \leq v_i$ . The  $k$  agents have *constant speeds*  $v_1, \dots, v_k$ , with starting points  $\beta_1, \dots, \beta_k$  when  $f_i(t) = v_i t + \beta_i \bmod \ell$  for all  $i = 1, \dots, k$ . A schedule is called *periodic* with period  $T > 0$  if  $f_i(t) = f_i(t + T) \bmod \ell$  for all  $i = 1, \dots, k$  and  $t \geq 0$ .  $H_n = \sum_{i=1}^n 1/i$  denotes the  $n$ th *harmonic number*; and  $H_0 = 0$ . If  $I$  is an interval,  $|I|$  denotes its length.

### 2 Track runners in the shade

We first show that the answer to the question posed in [7] is negative in general:

**Theorem 1** *Consider a circle  $C$  of unit length and a circular arc  $A \subset C$  of length  $\ell = |A| < 1$ . Then there exists  $k = k(\ell) \in \mathbb{N}$ , and a schedule for  $k$  runners with  $k$  distinct constant speeds and suitable starting points, so that at any time  $t \geq 0$ , at least one of the  $k$  runners is in the complement  $C \setminus A$ .*

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**Proof.** Set  $v_i = i$  as the speed of agent  $i$ , for  $i = 1, \dots, k$ , where  $k = k(\ell) \in \mathbb{N}$  will be specified later. Assume, as we may, that  $C \setminus A = [0, a]$ , for some  $a \in (0, 1)$ . Let  $t_0 = 0$ . Since the speed of each agent is an integer (and thereby multiple of the circle length  $\text{len}(C) = 1$ ), the resulting schedule is periodic and the period is 1. To ensure that at any  $t \geq 0$ , at least one agent is in  $[0, a]$ , it suffices to ensure this *covering condition* on the time interval  $[0, 1]$ , i.e., one period of the schedule. All agents start at time  $t = 0$ ; however, it is convenient to specify their schedule with their positions at a later time.

Agent 1 starts at point 0 at time 0; at time  $a$ , its position is at  $a$  (exiting  $[0, a]$ ). Agent 2 is at point 0 at time  $a$ ; at time  $a + a/2$ , its position is at  $a$  (exiting  $[0, a]$ ). Agent 3 is at point 0 at time  $a + a/2$ ; at time  $a + a/2 + a/3$ , its position is at  $a$  (exiting  $[0, a]$ ). Subsequent agents are scheduled according to this pattern. For  $i = 1, \dots, k$ , agent  $i$  is at point 0 at time  $aH_{i-1}$ ; at time  $aH_i$ , its position is at  $a$  (exiting  $[0, a]$ ). The schedules are given by the functions  $f_i(t) = it - iaH_{i-1}$  for  $i = 1, \dots, k$ .

The construction ensures that

1. agent  $i$  is in  $[0, a]$  during the time interval  $[aH_{i-1}, aH_i]$ , for  $i = 1, \dots, k$ .
2.  $\bigcup_{i=1}^k [aH_{i-1}, aH_i] \supseteq [0, 1]$ .

Indeed, condition 2 is  $aH_k \geq 1$ , or equivalently  $H_k \geq 1/a$ . Since  $\ln k \leq H_k$ , it suffices to have  $\ln k \geq 1/a$ , or  $k \geq \exp(1/a)$ , and the theorem is proved.  $\square$

The result extends to any finite number of circular arcs  $A_1, A_2, \dots, A_m \subset C$ . Stating the results for the complements  $B_i = C \setminus A_i$ , for  $i = 1, 2, \dots, m$ , we can schedule  $k$  mobile agents with distinct integer speeds so that at any time  $t \geq 0$ , each interval  $B_i$  contains at least one of the agents.

**Theorem 2** Consider a circle  $C$  of unit length and  $m$  circular arcs  $B_1, B_2, \dots, B_m \subset C$ , for some  $m \in \mathbb{N}$ . Then there exists  $k \in \mathbb{N}$ , and a schedule for  $k$  runners with  $k$  distinct constant speeds and suitable starting points, so that at any time  $t \geq 0$ , each of the arcs  $B_1, B_2, \dots, B_m$  contains at least one of the  $k$  runners.

**Proof.** In the proof of Theorem 1, we constructed a schedule of  $k(\ell)$  agents with speeds  $1, 2, \dots, k(\ell)$ . Note, however, that for any  $s \in \mathbb{N}$ , we could have used agents of speeds  $s + 1, s + 2, \dots, s + k(\ell, s)$ , such that

$$\bigcup_{i=s+1}^{s+k(\ell,s)} [aH_{i-1}, aH_i] \supseteq [0, 1]. \quad (1)$$

Indeed, for every  $s \in \mathbb{N}$  there exists  $k(\ell, s) \in \mathbb{N}$  satisfying (1), since  $\lim_{i \rightarrow \infty} H_i = \infty$ .

We can schedule  $k_1 = k(|B_1|, 0)$  agents with speeds  $1, 2, \dots, k_1$  such that at any time  $t \geq 0$ , the arc  $B_1 =$

$[0, a_1]$  contains at least one of these agents. For arc  $B_2$ , we can schedule  $k_2 = k(|B_2|, k_1)$  agents with speeds  $k_1 + 1, k_2 + 2, \dots, k_1 + k_2$  such that at any time  $t \geq 0$ , the arc  $B_2$  contains at least one of them. In general, if the first  $i - 1$  intervals are covered, let  $s_i = \sum_{j=1}^{i-1} k_j$ . Then we can schedule  $k_i = k(|B_i|, s_i)$  agents with speeds  $s_i + 1, s_i + 2, \dots, s_i + k_i$  such that at any time  $t \geq 0$ , the arc  $B_i$  contains at least one of them.  $\square$

Now that we have seen that the answer to our question is negative in general, it is however interesting to exhibit some scenarios (i.e., conditions) under which the answer is positive.

A set of real numbers  $\xi_1, \xi_2, \dots, \xi_k$  is said to be *rationaly independent* if no linear relation

$$a_1\xi_1 + a_2\xi_2 + \dots + a_k\xi_k = 0,$$

with integer coefficients, not all of which are zero, holds. In particular, if  $\xi_1, \xi_2, \dots, \xi_k$  are rationally independent, then they are pairwise distinct. Recall now Kronecker's theorem; see, e.g., [8, Theorem 444, p. 382]. (Although inessential, for conformity with the above formulation, our inequalities in Theorems 4 through 6 are strict.)

**Theorem 3** (Kronecker, 1884) If  $\xi_1, \xi_2, \dots, \xi_k \in \mathbb{R}$  are rationally independent,  $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$  are arbitrary, and  $T$  and  $\varepsilon$  are positive reals, then there is a real number  $t > T$ , and integers  $p_1, p_2, \dots, p_k$ , such that

$$|t\xi_m - p_m - \alpha_m| \leq \varepsilon \quad (m = 1, 2, \dots, k).$$

As a corollary, we obtain the following result.

**Theorem 4** Assume that  $k$  runners  $1, 2, \dots, k$ , with constant rationally independent (thus distinct) speeds  $\xi_1, \xi_2, \dots, \xi_k$ , run clockwise along a circle of length 1, starting from arbitrary points. For every circular arc  $A \subset C$  and for every  $T > 0$ , there exists  $t > T$  such that all runners are in  $A$  at time  $t$ .

**Proof.** Assume, as we may, that  $A = [0, a]$ , for some  $a \in (0, 1)$ . Let  $0 \leq \beta_i < 1$ , be the start position of runner  $i$ , for  $i = 1, 2, \dots, k$ . Set  $\alpha_i = a/2 + 1 - \beta_i$ , for  $i = 1, 2, \dots, k$ , set  $\varepsilon = a/3$ , and employ Theorem 3 to finish the proof.  $\square$

**Remark.** It is interesting to note that Theorem 1 gives a negative answer regardless of how long the shaded arc is, while Theorem 4 gives a positive answer regardless of how short the shaded arc is and for how far in the future one desires.

Observe that if  $\xi_1, \xi_2, \dots, \xi_k$  are rationally independent reals, then at least one  $\xi_i$  must be irrational (in fact, all but at most one  $\xi_i$  must be irrational). To obtain the conclusion of Theorem 4 neither the condition that the speeds  $\xi_1, \xi_2, \dots, \xi_k$  are rationally independent, nor the condition that at least one  $\xi_i$  is irrational

is necessary. For instance, by controlling the relative speeds allows one to obtain the same result with rational speeds, as shown in the following.

**Theorem 5** *Assume that  $k$  runners  $1, 2, \dots, k$ , with constant but distinct speeds run clockwise along a circle of length 1, starting from arbitrary points. For every circular arc  $A \subset C$ , there exist distinct rational speeds  $v_1, v_2, \dots, v_k > 0$ , so that for every  $T > 0$ , there exists  $t > T$  such that all runners are in  $A$  at time  $t$ .*

**Proof.** Assume, as we may, that  $[0, a] \subseteq A$ , for some rational  $a \in (0, 1)$ . Let  $\beta_1, \beta_2, \dots, \beta_k$  be the starting points of the runners, where  $0 \leq \beta_i < 1$ , for  $i = 1, 2, \dots, k$ . We proceed by induction on the number of runners  $k$ , and with a stronger induction hypothesis extending to every arc  $A$ . The base case  $k = 1$  is satisfied by setting  $v_1 = 1$  for any arc  $A$ . The subsequent speeds will be set to increasing values, so that  $v_1 < v_2 < \dots < v_k$ .

For the induction step, assume that the statement holds for runners  $1, 2, \dots, k - 1$ , the arc  $A' = [0, a/2]$  and  $T$ , and we need to prove it for runners  $1, 2, \dots, k$ , the arc  $A = [0, a]$  and  $T$ . By the induction hypothesis, there exists  $t > T$  so that runners  $1, 2, \dots, k - 1$ , are in  $A'$  at time  $t$ . Set  $v_k = \frac{2}{a}v_{k-1}$ ; since  $a, v_{k-1} \in \mathbb{Q}$ , we have  $v_k \in \mathbb{Q}$ . Observe that runner  $k$  will enter the arc  $A$  at point 0 before any of the first  $k - 1$  runners exits  $A$  at point  $a$ , regardless of his or her starting point. Hence all  $k$  runners will be in  $A$  at some time in the interval  $[t, t + 1/v_k]$ , completing the induction step, and thereby the proof of the theorem.  $\square$

In Theorem 5, the speeds  $v_1, v_2, \dots, v_k$  ensure that  $k$  runners are in a given circular arc  $A \subset C$  infinitely many times. Different intervals may require different speeds (based on the relative position of  $A$  and the  $k$  starting positions). The next theorem shows that, in fact, the same  $k$  speeds ensure this property for all circular arcs of a given length  $a > 0$ . Its proof is very similar to that of Theorem 5; for clarity we include both proofs.

**Theorem 6** *Assume that  $k$  runners  $1, 2, \dots, k$ , with constant but distinct speeds run clockwise along a circle of length 1, starting from arbitrary points. For every  $a \in (0, 1)$  there exist distinct rational speeds  $v_1, v_2, \dots, v_k > 0$ , so that for every  $T \geq 0$  and every circular arc  $A \subset C$  of length  $a$ , there exists  $t > T$  such that all runners are in  $A$  at time  $t$ .*

**Proof.** Let  $\beta_1, \beta_2, \dots, \beta_k$  be the starting points of the runners, where  $0 \leq \beta_i < 1$ , for  $i = 1, 2, \dots, k$ . We proceed by induction on the number of runners  $k$ . The base case  $k = 1$  is satisfied by setting  $v_1 = 1$  for any  $a > 0$ .

For the induction step, assume that the statement holds for  $k - 1$  runners, and we need to prove it for  $k$

runners. Let an arc length length  $a > 0$  and  $k$  starting positions  $\beta_1, \dots, \beta_k$  be given. By the induction hypothesis, for the arc length  $a' = a/2$  and  $k - 1$  starting points  $\beta_1, \dots, \beta_{k-1}$  there exist speeds  $v_1, \dots, v_{k-1}$  so that for any  $T \geq 0$  and any arc  $A' \subset C$  of length  $a' = a/2$ , all runners  $1, 2, \dots, k - 1$  are in  $A'$  at some time  $t > T$ .

Set  $v_k = \frac{2}{a}v_{k-1}$ . Consider an arbitrary arc  $A = [\alpha, \alpha + a] \subset C$  of length  $a$ . Denote the first half of the arc by  $A' = [\alpha, \alpha + a/2]$ . At time  $t$ , runners  $1, 2, \dots, k - 1$  are in  $A'$ . Observe that runner  $k$  will enter the arc  $A$  at point  $\alpha$  before any of the first  $k - 1$  runners exits  $A$  at point  $\alpha + a$ , regardless of his or her starting point. Hence all  $k$  runners will be in  $A$  at some time in the interval  $[t, t + 1/v_k]$ , completing the induction step, and thereby the proof of the theorem.  $\square$

The speeds of the agents in Theorems 5 and 6 can be chosen as integers if desired, by setting  $v_k = \lceil \frac{2}{a}v_{k-1} \rceil$ .

### 3 Conclusions and Open Problems

It is interesting to point out a connection between runners in the shade and idle time (as defined in Section 1). Assume that  $k$  runners  $1, 2, \dots, k$ , with constant rationally independent (thus distinct) speeds  $0 < \xi_1, \xi_2, \dots, \xi_k \leq 1$ , run clockwise along a circle of length 1, starting from arbitrary points. Further assume that  $\sum_{i=1}^k \xi_i = S$ , where  $S \leq k$  is large, say, close to  $k$ . A straightforward volume argument [5] yields the lower bound  $\tau \geq 1/\sum_{i=1}^k \xi_i = 1/S \geq 1/k$  on the idle time. On the other hand, by Theorem 3, for every circular arc  $A \subset C$  and for every  $T > 0$ , there exists  $t > T$  such that all runners are in  $A$  at time  $t$ ; pick an arbitrary interval  $A$  of length  $|A| = \varepsilon$ , where  $\varepsilon$  is small. Since the maximum speed of the agents is at most 1, the idle time  $\tau$  must be at least  $|C \setminus A| = 1 - \varepsilon$ . The example shows that the volume-based lower bound for the idle time can sometimes be very weak for large  $k$ .

The problems we have studied also suggest a few algorithmic questions for a circle  $C$  of unit length that we list below.

**Problem 1** *Given  $k$  runners with speeds  $v_1, \dots, v_k > 0$  and a circular arc  $A \subset C$ , decide whether there exist starting points  $\beta_1, \dots, \beta_k$  for the  $k$  runners, such that at any time  $t \geq 0$ , at least one of the runners is in  $A$ .*

Even if all runners start from the same point (say,  $\beta_i = 0$  for all  $i = 1, 2, \dots, k$ ), it is unclear how to test whether some runner will be in the shade at all times, or all runners will be out of the shade infinitely often.

**Problem 2** *Given  $k$  runners with speeds  $v_1, \dots, v_k > 0$  starting at 0 and a circular arc  $A \subset C$ , decide whether there exists  $T \geq 0$ , such that at any time  $t \geq T$ , at least one of the runners is in  $A$ .*

**Problem 3** Given  $k$  runners with speeds  $v_1, \dots, v_k > 0$  starting at 0, and an circular arc  $B \subset C$ , decide whether for any  $T \geq 0$  there exists  $t \geq T$  such that all  $k$  runners are in  $B$  at time  $t$ .

The following two problems are in some sense the “inverses” of Problem 1.

**Problem 4** Given  $k$  runners with speeds  $v_1, \dots, v_k > 0$  starting from points  $\beta_1, \dots, \beta_k \in C$ , respectively, and an arc length  $\ell > 0$ , decide whether there exist a circular arc  $A \subset C$  of length  $\ell$  and a time  $T \geq 0$  such that at any time  $t \geq T$  at least one of the runners is in  $A$ .

**Problem 5** Given  $k$  runners with starting points  $\beta_1, \dots, \beta_k \in C$ , a circular arc  $A \subset C$ , and a parameter  $v > 0$ , decide whether there exist rational speeds  $v_1, \dots, v_k \in (0, v)$  and a time  $T \geq 0$  such that at any time  $t \geq T$  at least one of the runners is in  $A$ .

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