Supplementary Material

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1. Proof of Theorem 1

Proof. **Step 1 – the "if" part.** Given a function $\rho(\mathbf{u}) = 1 - \sup\{k \in [0,1] | \sup_{\mathbf{v}\in\mathbf{V}_k}(-\mathbf{v}^\top\mathbf{u}) \le 0\}$ for some admissible class $\{\mathbf{V}_k\}$, we show that $\rho(\cdot)$ satisfies all properties required for a CCLF.

Step 1.1 – Complete Classification: If $\mathbf{u} \geq 0$, then by $\mathbf{V}_1 = \Re_m^+$ we have that $\mathbf{v}^\top\mathbf{u} \geq 0$ for all $\mathbf{v} \in \mathbf{V}_1$, which implies that $\sup_{\mathbf{v}\in\mathbf{V}_1}(-\mathbf{v}^\top\mathbf{u}) \le 0$. Hence $\rho(\mathbf{x}) = 0$. Conversely, if $\mathbf{u} \ngeq 0$, without loss of generality we assume $u_1 < 0$, then we have

$$\sup_{\mathbf{v}\in\mathbf{V}_1}(-\mathbf{v}^\top\mathbf{u}) = \sup_{\mathbf{v}\in\Re_+^m}(-\mathbf{v}^\top\mathbf{u}) \geq -\mathbf{e}_1\mathbf{u} > 0.$$

This, combined with $\mathbf{V}_1 = \mathrm{cl}(\lim_{k\uparrow 1}\mathbf{V}_k)$, leads to that $\exists\delta > 0$ such that

$$\sup_{\mathbf{v}\in\mathbf{V}_{1-\delta}}(-\mathbf{v}^\top\mathbf{u}) > 0,$$

which implies that $\rho(\mathbf{u}) > 0$. This shows that $\rho(\cdot)$ satisfies *complete classification*.

Step 1.2 – Misclassification avoidance: Fix $\mathbf{u}$ such that $\mathbf{u} < \mathbf{0}$. We have $\mathbf{e} \in \mathbf{V}_0$ which implies that

$$\sup_{\mathbf{v}\in\mathbf{V}_0}(-\mathbf{v}^\top\mathbf{u}) \geq (-\mathbf{e}^\top\mathbf{u}) > 0.$$

Hence $\rho(\mathbf{u}) = 1$. Thus, $\rho(\cdot)$ satisfies *misclassification avoidance*.

Step 1.3 – Monotonicity: If $\mathbf{u}_1 \le \mathbf{u}_2$, then for any $k \in [0,1]$, since $\mathbf{V}_k \subseteq \mathbf{V}_1 = \Re_+^m$, we have that $-\mathbf{v}^\top\mathbf{u}_1 \geq -\mathbf{v}^\top\mathbf{u}_2$ for any $\mathbf{v} \in \mathbf{V}_k$. Thus,

$$\{\sup_{\mathbf{v}\in\mathbf{V}_k}(-\mathbf{v}^\top\mathbf{u}_1) \le 0\} \quad \Longrightarrow \quad \{\sup_{\mathbf{v}\in\mathbf{V}_k}(-\mathbf{v}^\top\mathbf{u}_2) \le 0\}.$$

Hence $\rho(\mathbf{u}_1) \geq \rho(\mathbf{u}_2)$. Thus, $\rho(\cdot)$ satisfies *monotonicity*.

Step 1.4 – Order & scale invariance: Order invariance follows directly from the fact that $\mathbf{V}_k$ is order invariant for all k. Scale invariant holds because for $\alpha > 0$ and $k \in [0,1]$,

$$\{\sup_{\mathbf{v}\in\mathbf{V}_k}(-\mathbf{v}^\top\mathbf{u}) \le 0\} \quad \Longleftrightarrow \quad \{\sup_{\mathbf{v}\in\mathbf{V}_k}(-\mathbf{v}^\top\alpha\mathbf{u}) \le 0\}.$$

Step 1.5 – Quasi-convexity: To show quasi-convexity, let $c = \max(\rho(\mathbf{u}_1), \rho(\mathbf{u}_2))$ and without loss of generality assume $c < 1$ since otherwise the claim trivially holds. Thus we have that for any $\epsilon > 0$

$$\sup_{\mathbf{v}\in\mathbf{V}_{1-c-\epsilon}}(-\mathbf{v}^\top\mathbf{u}_i) \le 0, \quad i = 1, 2,$$

which implies that for $\alpha \in [0,1]$
$$\sup_{\mathbf{v}\in\mathbf{V}_{1-c-\epsilon}} \{-\mathbf{v}^\top[\alpha\mathbf{u}_1 + (1-\alpha)\mathbf{u}_2]\} \leq 0.$$
Thus $1-\rho(\alpha\mathbf{u}_1+(1-\alpha)\mathbf{u}_2) \geq 1-c$ since ϵ can be arbitrarily close to 0. The quasi-convexity holds.

Step 1.6 – Lower semi-continuity: We show that $\rho(\mathbf{u}_*) \leq \liminf_i \rho(\mathbf{u}_i)$ for $\mathbf{u}_i \xrightarrow{i} \mathbf{u}_*$. Let $c > \liminf_i \rho(\mathbf{u}_i)$, then there exists an infinite sub-sequence $\{\mathbf{u}_{i_j}\}$ such that $\rho(\mathbf{u}_{i_j}) < c$. That is
$$-\mathbf{v}^\top \mathbf{u}_{i_j} \leq 0; \quad \forall \mathbf{v} \in \mathbf{V}_{1-c}, \forall j.$$
Note that $\mathbf{u}_{i_j} \to \mathbf{u}_*$, hence
$$-\mathbf{v}^\top \mathbf{u}_* \leq 0; \quad \forall \mathbf{v}\in\mathbf{V}_{1-c},$$
i.e., $1-\rho(\mathbf{u}_*) \geq 1-c$. Since c can be arbitrarily close to $\liminf_i \rho(\mathbf{u}_i)$, the semi-continuity follows.

Step 2 – the "only if" part. Given a function $\rho(\cdot)$ which is a CCLF, we show that it can be represented as
$$\rho(\mathbf{u}) = 1-\sup\{k\in[0,1] | \sup_{\mathbf{v}\in\mathbf{V}_k}(-\mathbf{v}^\top\mathbf{u}) \leq 0\},$$
for some admissible class $\{\mathbf{V}_k\}$. This consists of three steps. We first show that $\rho(\cdot)$ can be represented as $\rho(\mathbf{u}) = 1-\sup\{k\in[0,1]|\sup_{\mathbf{v}\in\overline{\mathbf{V}}_k}(-\mathbf{v}^\top\mathbf{u}) \leq 0\}$, for some $\{\overline{\mathbf{V}}_k\}$. Here $\{\overline{\mathbf{V}}_k\}$ is not necessarily admissible, but satisfies $\overline{\mathbf{V}}_k \subseteq \overline{\mathbf{V}}_{k'}$ for all $k \leq k'$. We then show that we can replace $\overline{\mathbf{V}}_k$ by a class of closed, convex, order-invariant, cones $\mathbf{V}_k$. Finally we show that $\{\mathbf{V}_k\}$ is admissible to complete the proof.

Step 2.1. The representability of $\rho(\cdot)$ follows from Theorem 2 of (Brown & Sim, 2009). For completeness we re-state the result as a lemma, and provide the proof below.

Lemma A-1. *Given a CCLF $\rho(\cdot)$, then there exists $\{\overline{\mathbf{V}}_k\}$ that satisfies $\overline{\mathbf{V}}_k \subseteq \overline{\mathbf{V}}_{k'}$ for all $k\leq k'$, such that*
$$\rho(\mathbf{u}) = 1-\sup\{k\in[0,1] | \sup_{\mathbf{v}\in\overline{\mathbf{V}}_k}(-\mathbf{v}^\top\mathbf{u}) \leq 0\}.$$

Step 2.2. We construct $\{\mathbf{V}_k\}$ as follows. Let $\hat{\mathbf{V}}_k \triangleq \mathrm{cl}(\mathrm{cc}(\mathrm{or}(\overline{\mathbf{V}}_k)))$. Then we let $\mathbf{V}_k \triangleq \hat{\mathbf{V}}_k$ for $k\in(0,1)$, and $\mathbf{V}_0 \triangleq \bigcap_{k\in(0,1)} \hat{\mathbf{V}}_k$, and $\mathbf{V}_1 \triangleq \mathrm{cl}(\bigcup_{k\in(0,1)}\hat{\mathbf{V}}_k)$. Here $\mathrm{or}(\cdot)$ (respectively $\mathrm{cc}(\cdot)$) is the minimal **or**der invariant (respectively, **c**onvex **c**one) superset, defined as
$$\mathrm{or}(S) = \{P\mathbf{v}|P\in\mathcal{P}_n, \mathbf{v}\in S\}, \quad \mathrm{cc}(S) = \{\sum_{i=1}^k \lambda_i\mathbf{v}_i | k\in\mathbb{N}, \mathbf{v}_i\in S, \lambda_i\geq 0\}.$$
Let
$$\rho'(\mathbf{u}) = 1-\sup\{k\in[0,1] | \sup_{\mathbf{v}\in\hat{\mathbf{V}}_k}(-\mathbf{v}^\top\mathbf{u}) \leq 0\},$$
and observe that $\overline{\mathbf{V}}_k \subseteq \hat{\mathbf{V}}_k$, hence $\rho(\mathbf{u}) \leq \rho'(\mathbf{u})$. To show that $\rho(\mathbf{u}) \geq \rho'(\mathbf{u})$, it suffices to show that for any k, ϵ and $\mathbf{u}$, the following holds,
$$\{\sup_{\mathbf{v}\in\overline{\mathbf{V}}_k}(-\mathbf{v}^\top\mathbf{u}) \leq 0\} \implies \{\sup_{\mathbf{v}\in\hat{\mathbf{V}}_{k-\epsilon}}(-\mathbf{v}^\top\mathbf{u}) \leq 0\}. \tag{A-1}$$
Note that $\{\sup_{\mathbf{v}\in\overline{\mathbf{V}}_k}(-\mathbf{v}^\top\mathbf{u}) \leq 0\}$ implies $k \leq 1-\rho(\mathbf{u})$, and hence by order invariance of $\rho(\cdot)$, we have $k\leq 1-\rho(P\mathbf{u})$ for all $P\in\mathcal{P}_n$. This means
$$\sup_{\mathbf{v}\in\overline{\mathbf{V}}_{k-\epsilon}} \sup_{P\in\mathcal{P}_n}(-\mathbf{v}^\top P\mathbf{u}) \leq 0,$$
which is equivalent to
$$\sup_{\mathbf{v}\in\mathrm{or}(\overline{\mathbf{V}}_{k-\epsilon})}(-\mathbf{v}^\top\mathbf{u}) \leq 0.$$
By definition of $\mathrm{cc}(\cdot)$, this leads to
$$\sup_{\mathbf{v}\in\mathrm{cc}(\mathrm{or}(\overline{\mathbf{V}}_{k-\epsilon}))}(-\mathbf{v}^\top\mathbf{u}) \leq 0,$$

which further implies, by continuity of $-\mathbf{v}^\top\mathbf{u}$, that

$$\sup_{\mathbf{v}\in \mathrm{cl}(\mathrm{cc}(\mathrm{or}(\overline{\mathbf{V}}_{k-\epsilon})))} (-\mathbf{v}^\top\mathbf{u}) \leq 0.$$

Thus we have $\rho(\mathbf{u}) = \rho'(\mathbf{u})$. Finally note that $\hat{\mathbf{V}}_k \subseteq \hat{\mathbf{V}}_{k'}$ for $k \leq k'$, which leads to the following

$$\sup_{\mathbf{v}\in\hat{\mathbf{V}}_0} (-\mathbf{v}^\top\mathbf{u}) \leq \sup_{\mathbf{v}\in\bigcap_{k\in(0,1)}\hat{\mathbf{V}}_k} (-\mathbf{v}^\top\mathbf{u}) \leq \sup_{\mathbf{v}\in\hat{\mathbf{V}}_\epsilon} (-\mathbf{v}^\top\mathbf{u});$$

$$\sup_{\mathbf{v}\in\hat{\mathbf{V}}_{1-\epsilon}} (-\mathbf{v}^\top\mathbf{u}) \leq \sup_{\mathbf{v}\in\bigcup_{k\in(0,1)}\hat{\mathbf{V}}_k} (-\mathbf{v}^\top\mathbf{u}) \leq \sup_{\mathbf{v}\in\hat{\mathbf{V}}_1} (-\mathbf{v}^\top\mathbf{u}).$$

By definitions of $\mathbf{V}_0$ and $\mathbf{V}_1$, together with the fact (due to continuity)

$$\sup_{\mathbf{v}\in\mathrm{cl}(\bigcup_{k\in(0,1)}\hat{\mathbf{V}}_k)} (-\mathbf{v}^\top\mathbf{u}) = \sup_{\mathbf{v}\in\bigcup_{k\in(0,1)}\hat{\mathbf{V}}_k} (-\mathbf{v}^\top\mathbf{u}),$$

we conclude that

$$\rho(\mathbf{u}) = 1 - \sup\{k \in [0,1] | \sup_{\mathbf{v}\in\mathbf{V}_k} (-\mathbf{v}^\top\mathbf{u}) \leq 0\}.$$

Step 2.3. Now we check that $\{\mathbf{V}_k\}$ is indeed admissible. Property 1-3 are straightforward from the definition of $\mathbf{V}_k$. To see that $\mathbf{V}_0$ is closed, recall that the intersection of a class of closes sets is close.

We next show Property 4: $\mathbf{V}_1 = \Re^m_+$. By definition of $\mathbf{V}_1$, we have

$$\lim_{k\to 1} \sup_{\mathbf{v}\in\mathbf{V}_k} (-\mathbf{v}^\top\mathbf{u}) = \sup_{\mathbf{v}\in\mathbf{V}_1} (-\mathbf{v}^\top\mathbf{u}).$$

Hence $\rho(\mathbf{u}) = 0$ if and only if $\sup_{\mathbf{v}\in\mathbf{V}_1}(-\mathbf{v}^\top\mathbf{u}) \leq 0$. Thus by the property of *complete classification* we have the following

$$\{\sup_{\mathbf{v}\in\mathbf{V}_1} (-\mathbf{v}^\top\mathbf{u}) \leq 0\} \quad \Longleftrightarrow \quad \{\mathbf{u} \geq 0\}. \tag{A-2}$$

Denote the dual cone of a cone X by X^* and recall that for any k, $\mathbf{V}_k$ is a closed convex cone, hence we have

$$(\mathbf{V}_1^*)^* = \mathbf{V}_1.$$

The definition of dual cone states that

$$\mathbf{V}_1^* = \{\mathbf{u}|\mathbf{u}^\top\mathbf{v} \geq 0; \forall \mathbf{v} \in \mathbf{V}_1\},$$

which combined with Equation (A-2) implies that

$$\mathbf{V}_1^* = \Re^m_+.$$

Since $\Re^m_+$ is self-dual, we have

$$\mathbf{V}_1 = \Re^m_+.$$

We now turn to Property 5. Fix $k > 0$. Consider $\mathbf{u} = -\mathbf{e}$. By misclassification avoidance, $\rho(\mathbf{u}^*) = 1$, which means there exists $\mathbf{v}^* \in \mathbf{V}_k$ such that $\mathbf{v}^{*\top}\mathbf{u} < 0$, i.e., $\sum_{i=1}^m v_i > 0$. Define a permutation matrix $P \in \mathcal{P}_m$:

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, by order invariance of $\mathbf{V}_k$, $P^t\mathbf{v}^* \in \mathbf{V}_k$ for $t = 0, \cdots, m-1$. By convexity, this implies $\frac{1}{m}\sum_{t=0}^{m-1} P^t\mathbf{v}^* \in \mathbf{V}_k$. Note that $\frac{1}{m}\sum_{t=0}^{m-1} P^t\mathbf{v}^* = [\sum_{i=1}^m v_i^*]\mathbf{e}/m$, thus

$$\frac{\sum_{i=1}^m v_i^*}{m}\mathbf{e} \in \mathbf{V}_k.$$

Since $\sum_{i=1}^m v_i^* > 0$ and $\mathbf{V}_k$ is a cone, we have $\lambda\mathbf{e} \in \mathbf{V}_k$ for all $\lambda \geq 0$ and $k > 0$. By definition of $\mathbf{V}_0$, this implies $\lambda\mathbf{e} \in \mathbf{V}_0$. $\square$

The rest of this appendix provides a proof to Lemma A-1.

Proof. We recall the following results adapted from (Brown & Sim, 2009).

Definition A-1. *Let $\mathcal{U}$ be the set of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A function $\overline{\rho}(\cdot) : \mathcal{U} \to [0, 1]$ is a* collective satisfying measure *if the following holds for all $U, U' \in \mathcal{U}$.*

1. *If $U \geq 0$, then $\overline{\rho}(U) = 1$;*
2. *If $U < 0$, then $\overline{\rho}(U) = 0$;*
3. *If $U \geq U'$ then $\overline{\rho}(U) \geq \overline{\rho}(U')$;*
4. $\lim_{\alpha \geq 0} \overline{\rho}(U + \alpha) = \overline{\rho}(U)$*;*
5. *If $\lambda \in [0, 1]$, then $\overline{\rho}(\lambda U + (1 - \lambda)U') \geq \min(\overline{\rho}(U), \overline{\rho}(U'))$;*
6. *If $k > 0$, then $\overline{\rho}(kU) = \overline{\rho}(U)$.*

Theorem A-1. *Any collective satisfying measure $\overline{\rho}(\cdot)$ can be represented as*

$$\overline{\rho}(U) = \sup\{k \in [0, 1] | \sup_{\mathbb{Q} \in \mathcal{Q}_k} \mathbb{E}_{\mathbb{Q}}(-U) \leq 0\},$$

for a class of sets of probability measures $\mathcal{Q}_k$ satisfying $\mathcal{Q}_k \subseteq \mathcal{Q}_{k'}$ for $k \leq k'$.

Given this general result, we focus on a special case where $\Omega = \{1, 2 \cdots, m\}$. Note that in this case each random variable $U : \Omega \mapsto \Re$ can be represented as a vector $\mathbf{u} \in \Re^m$ where $u_i = U(i)$. Given a CCLF $\rho(\cdot) : \Re^m \to \Re$, we define $\overline{\rho} : \mathcal{U} \mapsto \Re$ as following

$$\overline{\rho}(U) = 1 - \rho(\mathbf{u}); \quad \text{where } u_i = U(i),\ i = 1, \cdots, m.$$

It is straightforward to check that $\overline{\rho}(\cdot)$ is a collective satisfying measure. Thus, Theorem A-1 states there exists a class of sets of probability measure $\mathcal{Q}_k$ such that

$$1 - \rho(\mathbf{u}) = \overline{\rho}(U) = \sup\{k \in [0, 1] | \sup_{\mathbb{Q} \in \mathcal{Q}_k} \mathbb{E}_{\mathbb{Q}}(-U) \leq 0\}.$$

Note that any probability measure Q on $\Omega = \{1, \cdots, m\}$ can be represented by a vector $\mathbf{v} \in \Re^m$ such that $v_i = Q(i)$. Thus $\mathbb{E}_Q(-X) = -\mathbf{v}^\top \mathbf{x}$ where $\mathbf{v}$ and $\mathbf{u}$ are the vector form for Q and U respectively. Hence we have there exists $\overline{\mathbf{V}}_k$ such that

$$\rho(\mathbf{u}) = 1 - \sup\{k \in [0, 1] | \sup_{\mathbf{v} \in \overline{\mathbf{V}}_k} (-\mathbf{v}^\top \mathbf{u}) \leq 0\}.$$

Note that for $k \leq k'$, $\overline{\mathbf{V}}_k \subseteq \overline{\mathbf{V}}_{k'}$ since $\mathcal{Q}_k \subseteq \mathcal{Q}_{k'}$. This concludes the proof of Lemma A-1. □

2. Proof of Theorem 2

Proof. **Claim 1:** We check that all conditions of Definition 1 are satisfied by $\overline{\rho}(\cdot)$. The only condition needs a proof is the semi-continuity. Consider a sequence $\mathbf{u}^j \to \mathbf{u}^0$, and let $t^0 = \max\{t : \sum_{i=1}^t u^0_{(i)} < 0\}$. Without loss of generality we let $u^0_1 \leq u^0_2 \leq, \cdots, \leq u^0_m$. Thus we have that $\sum_{i=1}^{t^0} u^0_i < 0$. This implies that $\limsup_j \sum_{i=1}^{t^0} u^j_i < 0$, which further leads to $\liminf_j (\max\{t : \sum_{i=1}^t u^j_{(i)} < 0\}) \geq t^0$. Hence $\liminf_j \overline{\rho}(\mathbf{u}^j) \geq \overline{\rho}(\mathbf{u}^0)$, which established the semi-continuity. Thus, we conclude that $\overline{\rho}(\cdot)$ is a CCLF. Further, observe that $\max\{t : \sum_{i=1}^t u_{(i)} < 0\} \geq \sum_{i=1}^m \mathbf{1}(u_i < 0)$, which established the first claim.

Claim 2: It is straightforward to check that $\overline{\mathbf{V}}_k$ satisfies all conditions of Definition 2, and hence is an admissible set. Thus, we proceed to show that $\overline{\mathbf{V}}_k$ is an admissible set *corresponding to* $\overline{\rho}(\cdot)$, i.e., to show

$$\overline{\rho}(\mathbf{u}) = 1 - \sup\{k \in [0, 1] | \sup_{\mathbf{v} \in \overline{\mathbf{V}}_k} (-\mathbf{v}^\top \mathbf{u}) \leq 0\}.$$

Fix a $\mathbf{u} \in \Re^m$. If $\mathbf{u} \geq 0$, then we have $\overline{\rho}(\mathbf{u}) = 0$, as well as $\sup_{\mathbf{v}\in\overline{\mathbf{V}}_1}(-\mathbf{v}^\top\mathbf{u}) \leq 0$, and hence the equivalence holds trivially. Thus we assume $\mathbf{u} \not\geq 0$, and let $t^0 = \max\{t : \sum_{i=1}^t u_{(i)} < 0\}$. By definition we have

$$\overline{\mathbf{V}}_{1-t^0/m} = \text{conv}\left\{\lambda\mathbf{e}_{N'} \,|\, \lambda > 0, |N'| = t^0 + 1\right\}.$$

Note that by definition of t^0

$$\min_{|N'|=t^0+1} \sum_{i\in N'} u_i \geq 0,$$

which implies that

$$\sup_{\mathbf{v}\in\{\mathbf{e}_{N'} | |N'|=t^0+1\}} (-\mathbf{v}^\top\mathbf{u}) \leq 0.$$

This leads to

$$\sup_{\mathbf{v}\in\overline{\mathbf{V}}_{1-t^0/m}} (-\mathbf{v}^\top\mathbf{u}) \leq 0. \tag{A-3}$$

On the other hand for arbitrarily small $\epsilon > 0$, by definition

$$\overline{\mathbf{V}}_{1-t^0/m+\epsilon} = \text{conv}\left\{\lambda\mathbf{e}_{N} \,|\, \lambda > 0, |N| = t^0\right\}.$$

Because $\min_{N:|N|=t^0} \sum_{i\in N} u_i < 0$, we have

$$\sup_{\mathbf{v}\in\overline{\mathbf{V}}_{1-t^0/m+\epsilon}} (-\mathbf{v}^\top\mathbf{u}) > 0.$$

Combining with Equation (A-3) we established the second claim.

Claim 3: Let $\rho'(\cdot)$ be a CCLF satisfying that $\rho'(\mathbf{u}) \geq \varrho(\mathbf{u})$ for all $\mathbf{u} \in \Re^m$, and let $\{\mathbf{V}'_k\}$ be its corresponding admissible set. Thus, it suffices to show that $\overline{\mathbf{V}}_k \subseteq \mathbf{V}'_k$ for all k. This holds trivially for $k = 0$, since $\rho'(\mathbf{u}) = 1$ for all $\mathbf{u} < \mathbf{0}$ implies that $\lambda\mathbf{e} \in \mathbf{V}'_0$. When $k > 0$, let $s/m < k \leq (s+1)/m$ for some integer s. Then, since $\mathbf{V}'_k$ is an order-invariant convex cone, it suffices to show that $\mathbf{e}_{[1:m-s]} \in \mathbf{V}'_k$ to establish the third claim. Consider $\mathbf{u}^* \triangleq -\mathbf{e}_{[1:m-s]}$. Then, by $\rho'(\mathbf{u}^*) \geq \sum_i \mathbf{1}(u^*_i < 0)/m = s/m < k$, we have

$$\begin{aligned} & \sup_{\mathbf{v}\in\mathbf{V}'_k} (-\mathbf{v}^\top\mathbf{u}^*) > 0 \\ \Longrightarrow \qquad & \exists \mathbf{v}^* \in \mathbf{V}'_k : \sum_{i=1}^{m-s} v^*_i > 0. \end{aligned}$$

Define a permutation matrix P:

$$P = \begin{bmatrix} P_1 & 0_{(m-s)\times s} \\ 0_{(m-s)\times s} & 0_{s\times s} \end{bmatrix},$$

where P_1 is a $(m-s)\times(m-s)$ matrix:

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, by order invariance of $\mathbf{V}'_k$, $P^t\mathbf{v}^* \in \mathbf{V}'_k$ for $t = 0, \cdots, m-s-1$. By convexity, this implies $\frac{1}{m-s}\sum_{t=0}^{m-s-1} P^t\mathbf{v}^* \in \mathbf{V}'_k$. Note that $\frac{1}{m-s}\sum_{t=0}^{m-s-1} P^t\mathbf{v}^* = [\sum_{i\in[1:m-s]} v^*_i]\mathbf{e}_{[1:m-s]}/(m-s)$, thus

$$\frac{\sum_{i=1}^{m-s} v^*_i}{m-s}\mathbf{e}_{[1:m-s]} \in \mathbf{V}'_k.$$

Since $\frac{\sum_{i=1}^{m-s} v^*_i}{m-s}$ is positive, and $\mathbf{V}'_k$ is a cone, we have $\mathbf{e}_{[1:n-s]} \in \mathbf{V}'_k$, which completes the proof. $\square$

3. Proof of Theorem 3

Proof. We prove the theorem by constructing such a function $\rho(\cdot)$. To do this, first consider $\breve{\rho} : \mathcal{R}^m \mapsto [0,1]$ defined as

$$\breve{\rho}(u) = \min_{\gamma > 0} \hat{\rho}(u/\gamma).$$

Then it is easy to check that $\breve{\rho}(\cdot)$ satisfies complete classification, misclassification avoidance, monotonicity, order invariance, and scale invariance. To see that $\breve{\rho}(u) \geq \varrho(u)$, note that if u has t negative coefficients, than for any $\gamma > 0$, u/γ also has t negative coefficients, which means

$$\hat{\rho}(u/\gamma) \geq t/m.$$

Taking minimization over γ, we have $\breve{\rho}(u) \geq \varrho(u)$ holds. Finally, we show quasi-convexity of $\breve{\rho}(\cdot)$. Fix u_1, u_2, and $\alpha \in [0,1]$, let γ_1, γ_2 be ϵ-optimal, i.e.,

$$\hat{\rho}(u_i/\gamma_i) \leq \breve{\rho}(u_i) + \epsilon, \quad i = 1, 2.$$

Since $\hat{\rho}$ is quasi-convex, we have

$$\begin{aligned}\hat{\rho}(\frac{\alpha u_1 + (1-\alpha)u_2}{\alpha\gamma_1 + (1-\alpha)\gamma_2}) &= \hat{\rho}(\frac{\alpha\gamma_1}{\alpha\gamma_1 + (1-\alpha)\gamma_2} \cdot \frac{u_1}{\gamma_1} + \frac{(1-\alpha)\gamma_2}{\alpha\gamma_1 + (1-\alpha)\gamma_2} \cdot \frac{u_2}{\gamma_2}) \\ &\leq \max\{\hat{\rho}(\frac{u_1}{\gamma_1}), \hat{\rho}(\frac{u_2}{\gamma_2})\}\end{aligned}$$

which implies

$$\breve{\rho}(\alpha u_1 + (1-\alpha)u_2) \leq \hat{\rho}(\frac{\alpha u_1 + (1-\alpha)u_2}{\alpha\gamma_1 + (1-\alpha)\gamma_2}) \leq \max\{\hat{\rho}(\frac{u_1}{\gamma_1}), \hat{\rho}(\frac{u_2}{\gamma_2})\} \leq \max\{\breve{\rho}(u_1), \breve{\rho}(u_2)\} + \epsilon.$$

Hence $\breve{\rho}(\cdot)$ is quasi-convex. Note that the only property that is not satisfied is the semi-continuity. To handle this, define $\rho : \mathcal{R}^m \mapsto [0,1]$ as

$$\rho(u) = \lim_{\epsilon \downarrow 0} \breve{\rho}(u + \epsilon e)$$

Because of monotonicity of $\breve{\rho}(\cdot)$, $\rho(\cdot)$ is well-defined. In addition, it can be shown that $\rho(\cdot)$ is lower-semicontinuous. Complete classification, misclassification avoidance, monotonicity, order invariance, scale invariance, and quasi-convexity all follows easily from the fact that same property holds for $\breve{\rho}(\cdot)$. Thus, $\rho(\cdot)$ is a CCLF w.r.t. m. Next, we show that

$$\hat{\rho}(u) \geq \rho(u) \geq \varrho(u).$$

The first inequality holds due to $\hat{\rho}(u) \geq \breve{\rho}(u) \geq \breve{\rho}(u + \epsilon e)$. The second inequality holds because for any u, there exists $\epsilon > 0$ small enough such that $\varrho(u + \epsilon e) = \varrho(u)$. Thus, taking limit over $\breve{\rho}(u + \epsilon e) \geq \varrho(u + \epsilon e)$ establishes the second inequality. Recall that $\bar{\rho}(u)$ is the minimal CCLF, we establish the lemma by

$$\varrho(u) \leq \bar{\rho}(u) \leq \rho(u).$$

□

4. Proof of Theorem 5

Proof. To prove Theorem 5, we start with establishing the following lemma. Observe that $\overline{\rho}(\mathbf{u})$ only takes value in $\{0, 1/m, 2/m, \cdots 1\}$.

Lemma A-2. *The level set of Problem (4), i.e.,* $\mathcal{U}_i \triangleq \{(\mathbf{u}, \mathbf{w}) | \overline{\rho}(\mathbf{u}) \leq 1 - i/m;\ f_j(\mathbf{u}, \mathbf{w}) \leq 0, \forall j\}$ *for* $i = 1, \cdots, m$, *equals the following*

$$\{(\mathbf{u}, \mathbf{w}) |\, \exists d : \sum_{i=1}^{m} [d - u_i]^+ \leq (m - i + 1)d;\ f_j(\mathbf{u}, \mathbf{w}) \leq 0,\ \forall j.\}$$

Proof. From Property 2 of Theorem 2, we have that $\mathcal{U}_i$ equals to the feasible set of the following program

$$\begin{aligned}&\sup_{\mathbf{v}\in\overline{\mathbf{V}}_{i/m}} (-\mathbf{v}^\top\mathbf{u}) \leq 0;\\ &f_j(\mathbf{u},\mathbf{w}) \leq 0; \quad j = 1,\cdots,n.\end{aligned}$$

Recall that $\overline{\mathbf{V}}_{i/m} = \mathrm{conv}\,\{\lambda\mathbf{e}_N \,|\, \lambda > 0, |N| = m-i+1\}$ we have that $\sup_{\mathbf{v}\in\overline{\mathbf{V}}_{i/m}}(-\mathbf{v}^\top\mathbf{u}) \leq 0$ is equivalent to

$$\inf_{\mathbf{v}:\mathbf{0}\leq\mathbf{v}\leq\mathbf{e},\mathbf{e}^\top\mathbf{v}=m-i+1} \mathbf{v}^\top\mathbf{u} \geq 0,$$

which left-hand-side by duality theorem is equivalent to the following optimization problem on $(\mathbf{c}, d)$

$$\begin{aligned}\text{Maximize:}\quad & \sum_{i=1}^m c_i + (m-i+1)d\\ \text{Subject to:}\quad & c_i + d \leq u_i\\ & c_i \leq 0.\end{aligned}$$

Thus we have $\mathbf{u}\in\mathcal{U}_i$ if and only if there exists $\mathbf{c}$, d, and $\mathbf{w}$ such that

$$\begin{aligned}&\mathbf{e}^\top\mathbf{c} + (m-i+1)d \geq 0;\\ &\mathbf{c} + d\mathbf{e} \leq \mathbf{u};\\ &\mathbf{c} \leq \mathbf{0};\\ &f_j(\mathbf{u},\mathbf{w}) \leq 0; \quad j = 1,\cdots,n.\end{aligned}$$

Note that this can be further simplified, since optimal $c_i = -[d-u_i]^+$, as

$$\begin{aligned}&\sum_{i=1}^m [d-u_i]^+ \leq (m-i+1)d\\ &f_j(\mathbf{u},\mathbf{w}) \leq 0; \quad j = 1,\cdots,n.\end{aligned} \tag{A-4}$$

This establishes the lemma. □

Now we turn to prove Theorem 5. Recall the assumption that there are no $\mathbf{u},\mathbf{w}$ such that $\mathbf{u}\geq 0$, and $f_j(\mathbf{u},\mathbf{w}) \leq 0$ for all j. Thus any feasible solution to (A-4) must have $d > 0$. Hence the feasible set to Problem (A-4) is equivalent to that of

$$\begin{aligned}&\sum_{i=1}^m [1-u_i/d]^+ \leq (m-i+1)\\ &f_j(\mathbf{u},\mathbf{w}) \leq 0; \quad j = 1,\cdots,m.\end{aligned}$$

Thus, finding the optimal solution to Problem (4) is equivalent to solve the following

$$\begin{aligned}\text{Minimize:}\quad & \sum_{i=1}^m [1-u_i/d]^+\\ \text{Subject to:}\quad & f_j(\mathbf{u},\mathbf{w}) \leq 0; \quad j = 1,\cdots,n;\\ & d > 0.\end{aligned} \tag{A-5}$$

By a change of variable where we let $h = 1/d$, $\mathbf{s} = \mathbf{u}h$, $\mathbf{t} = \mathbf{w}h$, this is equivalent to

$$\begin{aligned}\text{Minimize:}\quad & \sum_{i=1}^m [1-s_i]^+\\ \text{Subject to:}\quad & hf_j(\mathbf{s}/h,\mathbf{t}/h) \leq 0; \quad j = 1,\cdots,n;\\ & h > 0.\end{aligned}$$

Hence Theorem 5 is established. □

References

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