Three Hamilton Decomposition Problems

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Introduction

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A cycle in a graph is a connected subgraph in which each vertex has valency 2. A cycle containing every vertex of the graph is called a Hamilton cycle.

A Hamilton decomposition of a graph X is a partition of its edge set into Hamilton cycles, when X is regular of even valency, or a partition of its edge set into Hamilton cycles and a single perfect matching, when X is regular of odd valency.

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Theorem. (Ringel, 1954) *The n-dimensional cube Q_n has a Hamilton decomposition whenever n is a power of 2.*

History

Theorem. (Auerbach and Laskar, 1976) *The complete multipartite graph $K_{m;n}$ has a Hamilton decomposition whenever the valency is even.*

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The cartesian product $X \Box Y$ of two graphs X and Y is obtained by letting the vertex set be the cartesian product of the two vertex sets, and letting (u_1, v_1) be adjacent to (u_2, v_2) if and only if either

- $u_1 = u_2$ and v_1 and v_2 are adjacent in Y, or
- $v_1 = v_2$ and u_1 and u_2 are adjacent in X.

Three Problems

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The Cayley graph $\mathrm{Cay}(G; S)$ on the group G with connection set S may be thought of as the graph of order $|G|$ whose vertices are labelled with the elements of G such that g is adjacent with all vertices of the form gs as s runs over S. We insist that S is inverse-closed and $1 \notin S$.

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Problem. Does every connected Cayley graph on an abelian group have a Hamilton decomposition?

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We consider the third problem next.

Prisms Over Trivalent Graphs

If X is trivalent and has a cut-edge, then the prism over X has a 2-edge-cut. Clearly, $X \Box K_2$ is not Hamilton-decomposable because a Hamilton cycle must use a positive even number of edges from every edge-cut.

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If X is trivalent and has a cut-edge, then the prism over X has a 2-edge-cut. Clearly, $X \Box K_2$ is not Hamilton-decomposable because a Hamilton cycle must use a positive even number of edges from every edge-cut.

Thus, a necessary condition for $X \Box K_2$ to be Hamilton-decomposable is that X must be 2-edge-connected.

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Theorem. *If X is a trivalent graph with a perfect 1-factorization, then the prism over X has a Hamilton decomposition.*

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Theorem (Cada, Kaiser, Rosenfeld and Ryjaček, 2005). *The prism over a 3-connected, bipartite, planar trivalent graph has a Hamilton decomposition.*

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One application of their result is the establishment that the even dimensional cube has a Hamilton decomposition.

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Theorem (Stong, 1991). *Let X be regular of valency $2r$ and Y be regular of valency $2s$, where $r \leq s$. If both X and Y are Hamilton-decomposable, then $X \square Y$ is Hamilton-decomposable whenever one of the following holds:*
$s \leq 3r$, or
$r \geq 3$.

Cartesian Products

The glaring case not covered by Stong's Theorem is the cartesian product of a Hamilton-decomposable graph of small valency with a Hamilton-decomposable graph of large valency. In particular, his theorem does not cover the case of the cartesian product of a single cycle with a Hamilton-decomposable graph.

Cayley Graphs On Abelian Groups

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If the graph has valency 3, then it is well known that it has a Hamilton cycle. Removing the Hamilton cycle must leave a perfect matching and the graph is Hamilton-decomposable.

Theorem (Bermond, Favaron and Maheo, 1989) *If X is a connected Cayley graph of valency 4 on an abelian group, then X is Hamilton-decomposable.*

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Let's consider a Cayley graph X of valency 5 on an abelian group. The connection set S must contain an involution s.

Let X' be the Cayley graph on the same group with connection set $S - s$. Clearly, X' is a subgraph of X.

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Let X' be the Cayley graph on the same group with connection set $S - s$. Clearly, X' is a subgraph of X.

If X' is connected, then it has a decomposition into two Hamilton cycles by the Bermond-Favaron-Maheo Theorem. The element s generates a perfect matching yielding a Hamilton decomosition of X.

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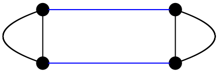

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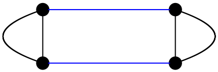

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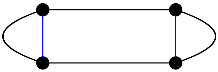

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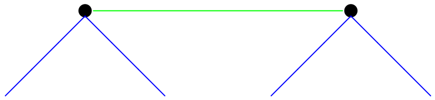

Therefore, a connected Cayley graph of valency 5 on an abelian group has a Hamilton decomposition.

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Theorem (Westland, Liu, and Kreher, 2009). *If X is a connected Cayley graph of valency 6 on an odd order abelian group, then X is Hamilton-decomposable.*

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The following result has been proven but the paper is not out yet.

Theorem (Westland, Liu, and Kreher, 2009). *If X is a connected Cayley graph of valency 6 on an odd order abelian group, then X is Hamilton-decomposable.*

That is as far as people have progressed in the direction of restricting the valency.

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If S is a subset of a group, where $1 \notin S$, let $\bar{\mathrm{I}}(S)$ denote the *inverse-closure* of S, that is, the smallest superset of S that is inverse-closed.

Theorem (Liu, 1996 and 2003) *If S is a minimal generating set for the abelian group G, then the Cayley graph $\mathrm{Cay}(G; \bar{\mathrm{I}}(S))$ is Hamilton-decomposable.*

Cayley Graphs On Abelian Groups

Note that Liu's Theorem includes the earlier results we mentioned about cartesian products of cycles because they are Cayley graphs on direct products of cyclic groups, where the connection set consists of the standard basis and their inverses.

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We now move to a new result that has been submitted to the special issue of *Discrete Mathematics* celebrating Gert Sabidussi's 80th birthday. We discuss a few preliminary notions first.

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An element x of a field $\mathbb{F}$ is a quadratic residue if there is a $y \in \mathbb{F}$ for which $y^2 = x$. Let $\mathbb{F}(q)$ denote the finite field of order q.

It is well known that the quadratic residues of $\mathbb{F}(q)$ form a multiplicative subgroup of $\mathbb{F}(q)^*$ of order $(q-1)/2$, where $\mathbb{F}(q)^*$ denotes $\mathbb{F}(q)$ with the element 0 removed. The latter is a multiplicative group of order $q-1$.

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The Paley graph $\mathbb{P}(q)$, $q \equiv 1 (\text{mod } 4)$, has the elements of the field $\mathbb{F}(q)$ for its vertex set, and x and y are adjacent if and only if $x - y$ is a quadratic residue in $\mathbb{F}(q)$.

Paley Graphs

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So it apparently is the case that no one had settled the problem of whether Paley graphs are Hamilton-decomposable.

Paley Graphs

Theorem (Alspach, Bryant, and Dyer, 2010) *The Paley graphs have Hamilton decompositions.*

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Here is a rough discussion of the proof.

Paley Graphs

Let $q = p^e$ with $q \equiv 1 (\text{mod } 4)$. The first important feature to observe about the Paley graph $\mathbb{P}(q)$ is that the edges of the graph are generated by the additive structure of the field. Because the characteristic of the field is p, every element of the connection set generates a 2-factor in which each component is a cycle of length p.

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If y is a scalar multiple of x, then by considering the cycle $0, x, 2x, 3x, \ldots, (p-1)x$ contained in F, we see that y generates edges inside the components of F. The element y does not connect any of the components of F to each other.

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The converse also holds in that if the element y generates edges inside the components of F, then y must be a scalar multiple of x.

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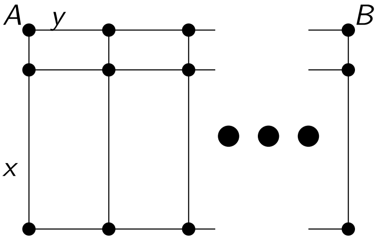

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Lemma *If $x_1, x_2, \ldots, x_t$ are linearly independent, then the components of the subgraph generated by $x_1, x_2, \ldots, x_t$ are isomorphic to the cartesian product of t p-cycles.*

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Corollary *If $x_1, x_2, \ldots, x_e$ form a basis for $\mathbb{F}(q)$, then $\mathrm{Cay}(\mathbb{F}(q); \bar{I}(x_1, x_2, \ldots, x_e))$ is Hamilton-decomposable.*

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Proof. The graph is isomorphic to a cartesian product of e p-cycles so that the Cayley graph is connected. It is Hamilton-decomposable by an earlier lemma.

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The preceding corollary gives us an obvious strategy to prove that all Paley graphs have Hamilton decompositions. Simply partition the quadratic residues into bases. However, it isn't that simple. First of all, we don't want to partition all of the quadratic residues because when we build the connected Cayley subgraphs we include the additive inverses as well.

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So there are $(q-1)/4$ quadratic residues to partition into bases, that is, sets of cardinality e each of which is linearly independent. This brings up the next problem: e may not divide $(q-1)/4$.

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For each pair $\pm x$ of quadratic residues, choose precisely one of them to place in a set B. Thus, $|B| = (q-1)/4$ and $\bar{I}(B) = QR(q)$. Note that $\mathrm{Cay}(\mathbb{F}(q); \bar{I}(B))$ is the Paley graph $\mathbb{P}(q)$.

Paley Graphs

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Theorem. (Edmonds-Fulkerson, 1965) *Let M be a matroid of cardinality n with rank function ρ. If $n_1, n_2, \ldots, n_t$ are positive integers satisfying $\sum_{i=1}^{t} n_i = n$, then M can be partitioned into independent sets of cardinalities $n_1, n_2, \ldots, n_t$ if and only if $n_i \leq \rho(M)$ for all i, and for every subset $S \subseteq M$ we have*

$$|S| \leq \sum_{i=1}^{t} min(n_i, \rho(S)).$$

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where $0 \leq r < e$.

It is not difficult to apply the Edmonds-Fulkerson Theorem to obtain that B may be partitioned into α bases (linearly independent sets of cardinality e) and a linearly independent set of cardinality r. Of course, the bases give us subgraphs of $\mathbb{P}(q)$ that are Hamilton-decomposable, but what do we do with the r vectors left over? The subgraph they generate isn't even connected.

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Place x with cx so that in the cartesian product arising from B_i, the elements x and cx produce a 4-valent component. Do a similar thing for each element of B_0.

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Place x with cx so that in the cartesian product arising from B_i, the elements x and cx produce a 4-valent component. Do a similar thing for each element of B_0.

Because cartesian products are associative and commutative, we use Stong's Theorem to prove that the resulting product is Hamilton-decomposable.

Paley Graphs

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Because cartesian products are associative and commutative, we use Stong's Theorem to prove that the resulting product is Hamilton-decomposable.

For the primes 3 and 5 we use several tricks to use the left over vectors. We shall not do so here.

Thank You