Towards a Unified Method for Termination

Akihisa Yamada

ERATO HASUO Metamathematics for Systems Design Project

NII, Tokyo

Contents

- abstract things
- monotone WPO (weighted path order)
- weakly monotone WPO
- constrained WPO
- non-monotone WPO

Abstract reduction system (ARS), Klop-style

is an indexed family $\{\to_\rho \subseteq T \times T\}_{\rho \in R}$ Write $\to_R := \bigcup_{\rho \in R} \to_\rho$

- **Term rewrite system (TRS)**:

  $T = \mathcal{T}(F,V) \qquad \rho \in \mathcal{T}(F,V) \times \mathcal{T}(F,V) \qquad \to_\rho := \underset{\rho}{\longrightarrow}$

- **DP problem**:

  $T = SN\left(\overset{>\epsilon}{\underset{\mathcal{R}}{\longrightarrow}}\right) \qquad \rho \in \mathcal{T}(F,V) \times \mathcal{T}(F,V) \qquad \to_\rho := \overset{\epsilon}{\underset{\rho}{\longrightarrow}} \circ \overset{>\epsilon}{\underset{\mathcal{R}}{\longrightarrow}}{}^*$

- **Integer transition system (ITS)**:

  $T = L \times \mathbb{Z}^V \qquad \rho \in L \times \mathcal{T}_{bool}(\mathcal{Z}, V \uplus V) \times L$

  $$\frac{[\![\phi]\!](\alpha \uplus \beta) = \mathrm{True}}{\langle l, \alpha \rangle \xrightarrow[\langle l,\phi,r \rangle]{} \langle r, \beta \rangle}$$

- **Constrained TRS**:

  $T = \mathcal{T}(F,V) \qquad \rho \in T \times \mathcal{T}_{bool}(\mathcal{Z}, V) \times T$

  $$\frac{[\![\phi\theta]\!] = \mathrm{True}}{l\theta \xrightarrow[\langle l,\phi,r \rangle]{} r\theta}$$

Termination of ARSs

- $\to_R$ is ***terminating*** if no $\to_R \circ \to_R \circ \to_R \cdots$
- Let $f : T \to A$, $\langle A, \geq, > \rangle$ well-founded
- Define $[\geq]^f_\to := \{\, \rho \mid s \to_\rho t \implies f(s) \geq f(t) \,\}$

For reduction order $\geq$, $f = id$ and $[\geq] = \geq$

Proposition:

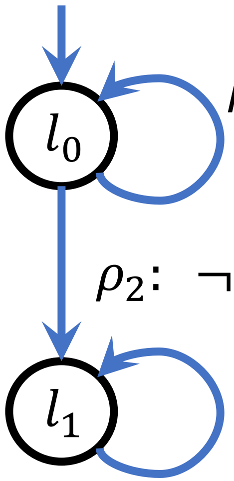
If $R \subseteq [\geq]$, then $\to_R$ is terminating Iff $\to_{R \setminus [>]}$ is.

In case of ITS…

Example (in C):

```
int x, y;
while (x <= 0) {
  x = x + 2;
  y = y + 4;
}
while (y >= -3) {
  x = 2 * x;
  y = y - x;
}
```

Example (in ITS):



ρ_1: $x \leq 0 \wedge x' = x + 2 \wedge y' = y + 4$

ρ_2: $\neg\, x \leq 0 \wedge x' = x \wedge y' = y$

ρ_3: $y \geq -3 \wedge x' = 2x \wedge y' = y - x'$

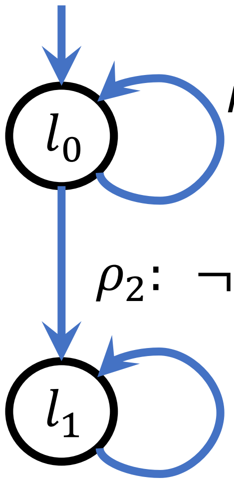
- $f : L \times \mathbb{Z}^V \to A$, $\langle A, \geq, > \rangle$ well-founded

In case of ITS...

Example (in C):

```
int x, y;
while (x <= 0) {
  x = x + 2;
  y = y + 4;
}
while (y >= -3) {
  x = 2 * x;
  y = y - x;
}
```

Example (in ITS):



ρ_1: $x \leq 0 \wedge x' = x + 2 \wedge y' = y + 4$

ρ_2: $\neg\, x \leq 0 \wedge x' = x \wedge y' = y$

ρ_3: $y \geq -3 \wedge x' = 2x \wedge y' = y - x'$

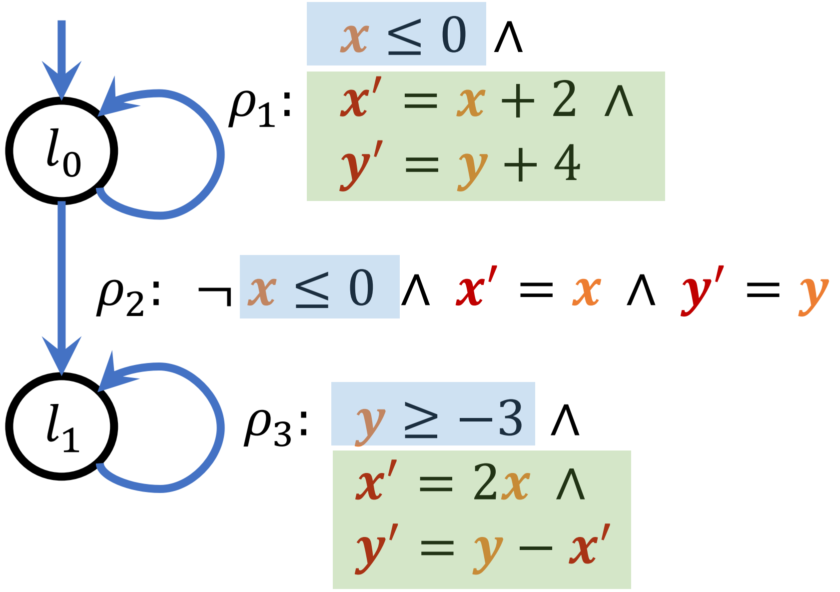
- $f : L \to \mathbb{Z}^V \to A$, $\langle A, \geq, > \rangle$ well-founded

In case of ITS...

Example (in C):

```
int x, y;
while (x <= 0) {
  x = x + 2;
  y = y + 4;
}
while (y >= -3) {
  x = 2 * x;
  y = y - x;
}
```

Example (in ITS):



- $f : L \to \mathbb{Z}^V \to \mathbb{Z}$, $\langle \mathbb{Z}, \geq, >_b \rangle$ well-founded

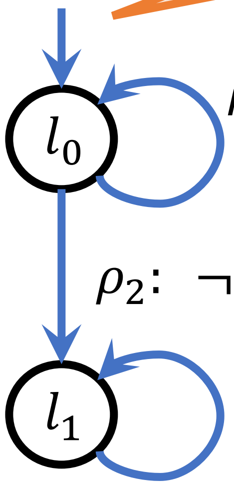
In case of ITS…

but start location is also important

Example (in C):

```
int x, y;
while (x <= 0) {
  x = x + 2;
  y = y + 4;
}
while (y >= -3) {
  x = 2 * x;
  y = y - x;
}
```

Example (in ITS):



ρ_1: $x \leq 0 \land x' = x + 2 \land y' = y + 4$

ρ_2: $\neg\, x \leq 0 \land x' = x \land y' = y$

ρ_3: $y \geq -3 \land x' = 2x \land y' = y - x'$

- $f : L \to T(\mathcal{Z}, V)$, $\langle \mathbb{Z}, \geq, >_b \rangle$ well-founded
  - $\langle l, \phi, r \rangle \in [\geq] \iff \vDash_{\mathcal{Z}} \phi \Rightarrow f(l) \geq f(r)'$
  - $\langle l, \phi, r \rangle \in [>] \iff \vDash_{\mathcal{Z}} \phi \Rightarrow f(l) > f(r)' \land f(l) \geq b$

Local termination [Endrullis+'10]

- $\to_R$ is ***terminating on*** S if no $S \ni \circ \to_R \circ \to_R \circ \to_R \cdots$
- Not incremental, in the sense...

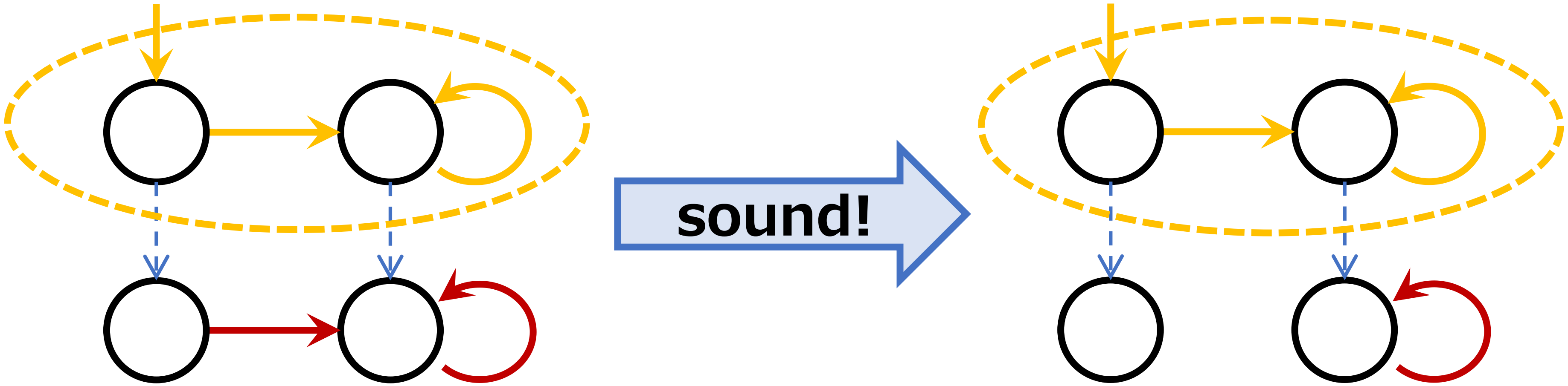
Recurrence [Brockschmidt+'13?], generalized

- **Definition**:
  - $s \circlearrowleft_C t$ iff $s \to_{\rho_1} \cdots \to_{\rho_n} t$ with $\{\rho_1, \ldots, \rho_n\} = C$
  - P is ***nonrecurrent on*** $S :\Leftrightarrow \forall C \subseteq P.\ \circlearrowleft_C$ is terminating on S

reachability (out of scope)

Proposition:

For finite P, $\to_P$ is terminating on S iff P is nonrecurrent on $\to_P^* (S)$.

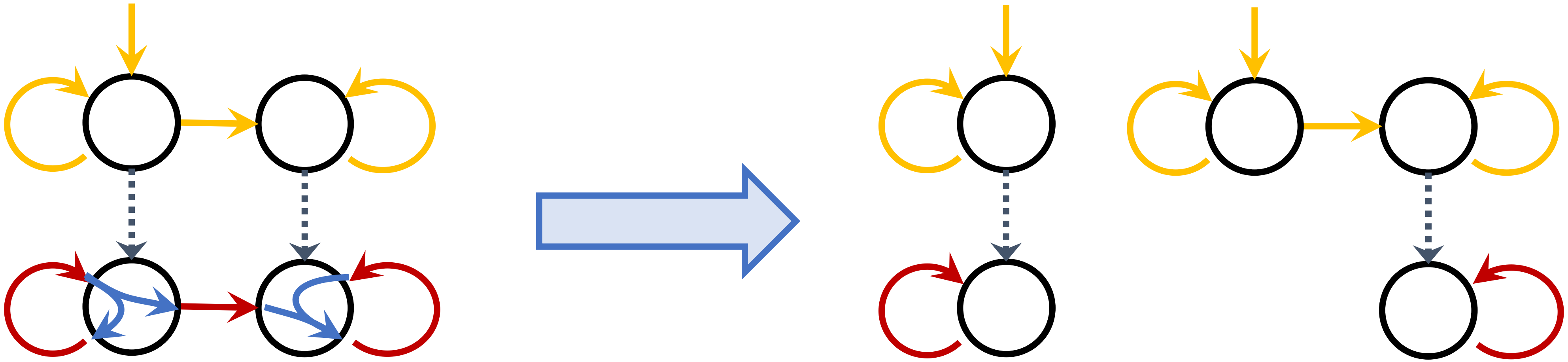


SCC decomposition, generalized

- $\mathbf{DG}_{\rightarrow}(P) := \langle P, \{ \langle \rho, \rho' \rangle \mid \rightarrow_{\rho} \circ \rightarrow_{\rho'} \neq \emptyset \} \rangle$

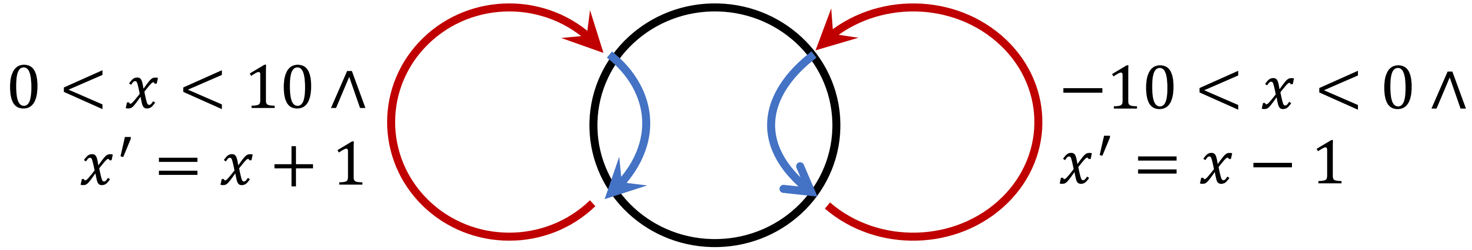
Proposition

P is nonrecurrent on S if every SCC of $\mathrm{DG}(P)$ is.



Generalizes standard SCC technique for ITSs, in the sense...

An artificial example



DG for local termination

pick minimal nonterminating subterm

future work

- $\underset{\mathcal{R}}{\to}$ is terminating on S

  $\iff \xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon} \circ \underset{\mathcal{R}}{\to}^{*}$ is terminating on $\trianglerighteq(S)$

  $\iff$ (if $\mathrm{DP}(\mathcal{R})$ is finite) $\mathrm{DP}(\mathcal{R})$ is nonrecurrent on $\trianglerighteq \circ \left(\xrightarrow[\mathrm{DP}(\mathcal{R})]{\epsilon} \circ \underset{\mathcal{R}}{\to}^{*} \right)^{*} (S)$

Corollary (local dependency graph):

If $\mathcal{P}$ is finite, then $\overset{\epsilon}{\underset{\mathcal{P}}{\to}} \circ \xrightarrow[\mathcal{R}]{>\epsilon}$ is nonrecurrent on S if $\overset{\epsilon}{\underset{\mathcal{C}}{\to}} \circ \xrightarrow[\mathcal{R}]{>\epsilon}$ for every SCC $\mathcal{C}$ of $\mathrm{DG}(\mathcal{P})$ is.

$$\mathrm{DG}(\mathcal{P}) = \left\langle \mathcal{P}, \left\{ \langle l \to r, l' \to r' \rangle \;\middle|\; \exists \theta, \theta'.\ r\theta \overset{>\epsilon}{\underset{\mathcal{R}}{\to}} l'\theta' \right\} \right\rangle$$

Contents

- abstract things
- **monotone WPO (weighted path order)**
- weakly monotone WPO
- constrained WPO
- non-monotone WPO

Monotone interpretations

- **Well-founded $\mathcal{F}$-algebra** $[\![\cdot]\!]$ assigns
  - each $f \in \mathcal{F}_n$, $[\![f]\!] : A^n \to A$
  - $\langle A, \geq, > \rangle$, well-founded ordering,
- Define $[\![\cdot]\!] : T(\mathcal{F}, V) \to (V \to A) \to A$ by
  - $[\![x]\!]\alpha = \alpha(x)$
  - $[\![f(s_1, \dots, s_n)]\!]\alpha = [\![f]\!]([\![s_1]\!]\alpha, \dots, [\![s_n]\!]\alpha)$
  - $[\![s \geq t]\!]\alpha :\Leftrightarrow [\![s]\!]\alpha \geq [\![t]\!]\alpha$
- $[\![\cdot]\!]$ is monotone if every $[\![f]\!]$ is monotone in every argument

Theorem:

A TRS $\mathcal{R}$ is terminating iff there is a w.f. monotone algebra $[\![\cdot]\!]$ s.t. $\forall l \to r \in \mathcal{R}. [\![l > r]\!]$

Reduction orders

- Well-founded term algebra $[\![\cdot]\!]$:
  - $[\![f]\!]\,(s_1, \ldots, s_n) = f(s_1, \ldots, s_n)$
  - $\langle T(F, V), \geq, > \rangle$, well-founded ordering
- $[\![\cdot]\!] : T(F,V) \to \big(V \to T(F,V)\big) \to T(F,V)$ is just substitution
  - $[\![x]\!]\theta = \theta(x)$
  - $[\![f(s_1, \ldots, s_n)]\!]\theta = [\![f]\!]([\![s_1]\!]\theta, \ldots, [\![s_n]\!]\theta)$
  - $[\![s \geq t]\!]\theta :\Leftrightarrow [\![s]\!]\theta \geq [\![t]\!]\theta$
- $[\![\cdot]\!]$ is monotone if every f is monotone in every argument

Theorem:

A TRS $\mathcal{R}$ is terminating iff there is a reduction order $>$ s.t. $\mathcal{R} \subseteq >$

Simplification Order [Dershowitz '82]

- A rewrite order satisfying **subterm property**: $f(\ldots x \ldots) \succ x$

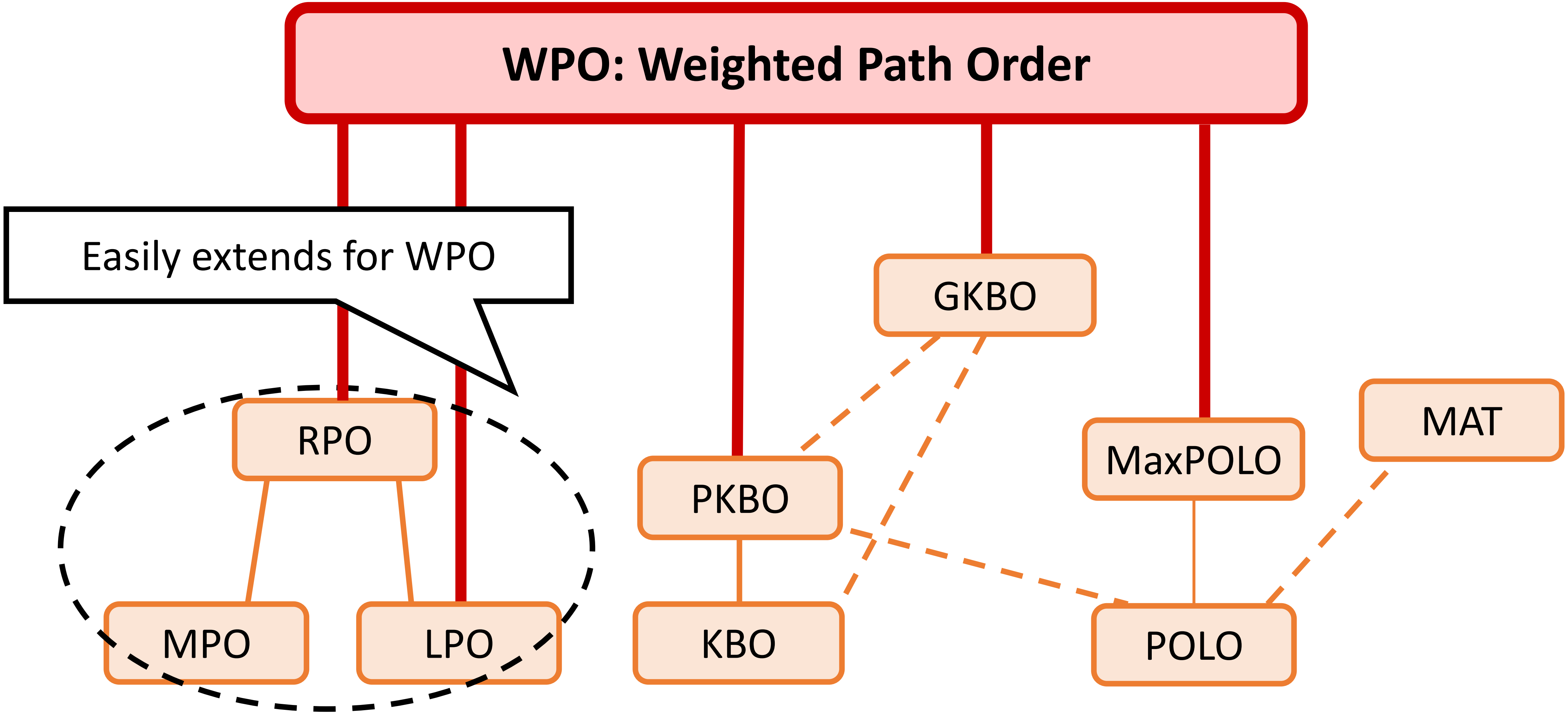
Theorem [Dershowitz '82]

For finite $\mathcal{F}$, a simplification order is a reduction order

(Some) existing monotone methods

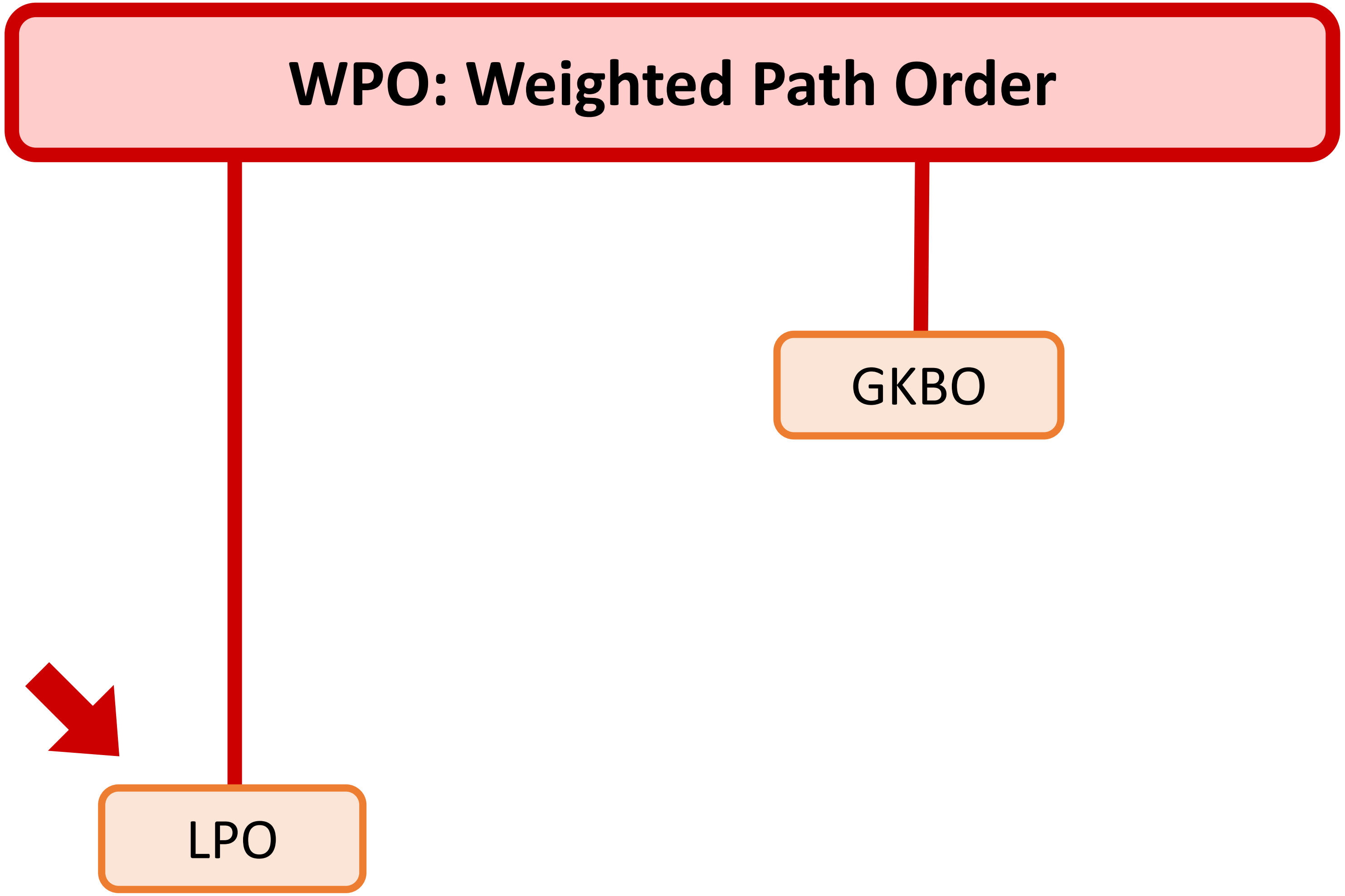
- **KBO**: Knuth-Bendix Order [Knuth & Bendix '70]
- **POLO**: Polynomial Order [Manna & Ness '70, Lankford '75]
- **PKBO**: Polynomial KBO [Lankford '79]
- **MPO**: Multiset Path Order [Dershowitz '79, '82]
- **LPO**: Lexicographic Path Order [Kamin & Lévy '80]
- **RPO**: Recursive Path Order [Lescanne '83]
- **GKBO**: Generalized KBO [Middeldorp & Zantema '97]
- **TKBO**: Transfinite KBO [Ludwig & Waldmann '07]
- **MAT**: Matrix Interpretations [Endrullis+ '08]

Map



WPO: Weighted Path Order
Easily extends for WPO
GKBO
MAT
RPO
PKBO
MaxPOLO
MPO
LPO
KBO
POLO

Map



LPO [Kamin & Lévy '80]

- **Definition:**

  $s = f(s_1, \ldots, s_n) \succ_{\text{LPO}} t$ iff

  a. $\exists i. s_i \succsim_{\text{LPO}} t$ or

  b. $\forall j. s \succ_{\text{LPO}} t_j$ for $t = g(t_1, \ldots, t_m)$ and

  ensure subterm property

  i. $f \succ g$ or

  ii. $f \sim g$ and $[s_1, \ldots, s_n] \succ_{\text{LPO}}^{\text{lex}} [t_1, \ldots, t_m]$

  ensure monotonicity

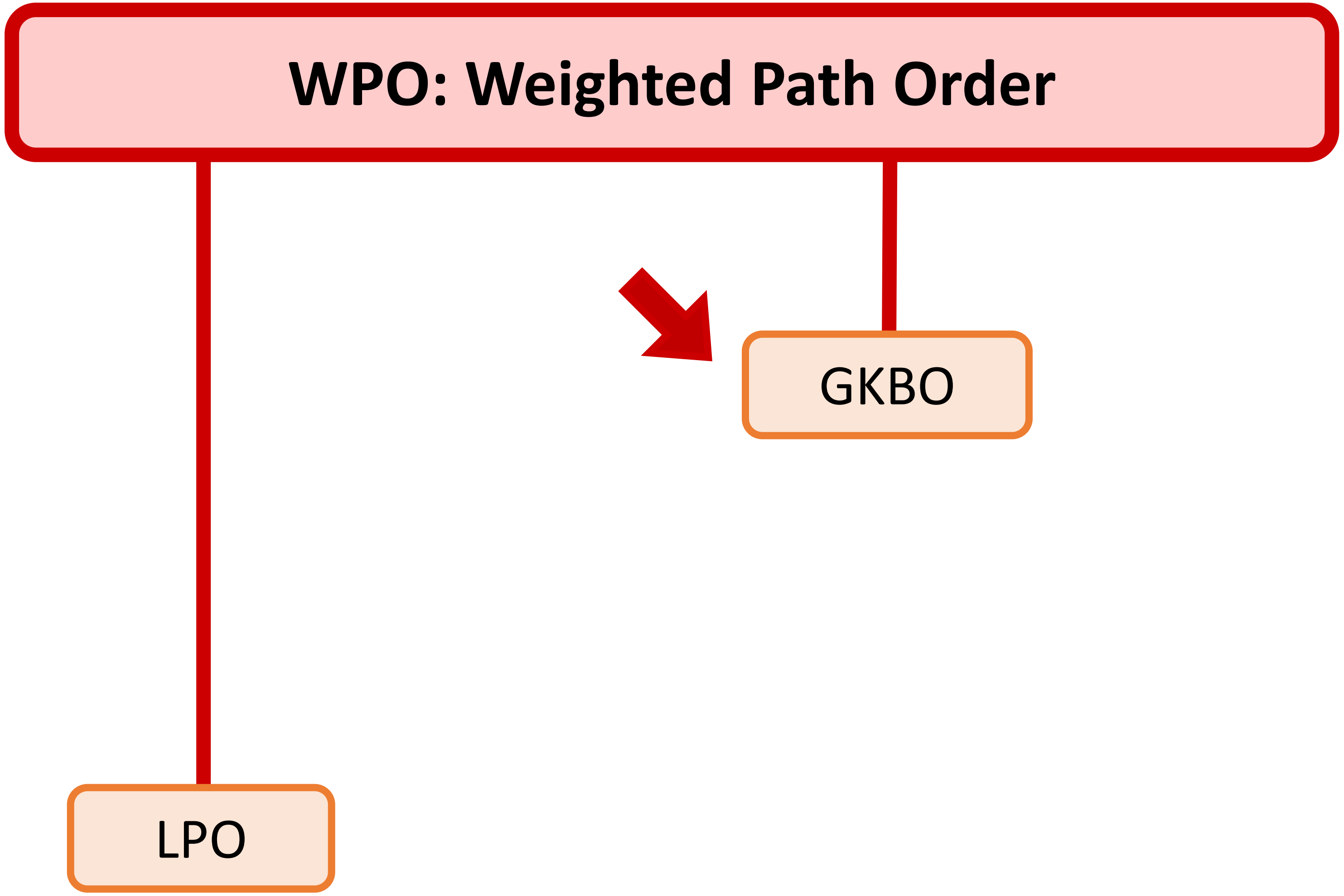
- **Example:** $\text{nil} \succ 0 \succ \text{sum} \succ +$

  $\text{sum}(\underline{\text{nil}}) \succ_{\text{LPO}} \underline{0}$

  $\underline{\text{sum}}(x :: xs) \succ_{\text{LPO}} x \mathbin{\underline{+}} \text{sum}(xs)$

Theorem [Kamin & Lévy '80]: $\succ_{\text{LPO}}$ is a simplification order.

Map



GKBO [Middeldorp & Zantema '97]

- In a sketch...
  - compare ***weights*** defined by a well-founded F-algebra $[\![\cdot]\!]$
  - then compare ***precedence*** $\succ$ (like LPO)
- Example

$$\begin{cases} [\![\mathrm{sum}]\!](x) \mapsto x + 1 \\ [\![\mathrm{nil}]\!] \mapsto 0 \\ [\![0]\!] \mapsto 0 \\ x \mathbin{[\![::]\!]} y \mapsto x + y + 1 \\ x \mathbin{[\![+]\!]} y \mapsto \max(x, y) + 1 \end{cases}$$

$$\mathrm{sum} \succ +$$

$$\overbrace{\mathrm{sum}(\mathrm{nil})}^{\mathbf{1}} \overset{>}{\succ_{\mathrm{GKBO}}} \overbrace{0}^{\mathbf{0}}$$

$$\underbrace{\underline{\mathrm{sum}}(x :: xs)}_{x + xs + \mathbf{2}} \underset{\geq}{\overset{}{\succ_{\mathrm{GKBO}}}} \underbrace{x \mathbin{\underline{+}} \mathrm{sum}(xs)}_{\max(x, xs + 1) + 1} \quad (\mathrm{sum} \succ +)$$

GKBO [Middeldorp & Zantema '97]

- **Definition**:

  $s = f(s_1, \dots, s_n) \succ_{\mathrm{GKBO}} t$ iff

  1. $[\![s > t]\!]$ or
  2. $[\![s \geq t]\!]$ for $t = g(t_1, \dots, t_n)$ and

  compare weights

     i. $f \succ g$ or

     ii. $f \sim g$ and $[s_1, \dots, s_n] \succ^{\mathrm{lex}}_{\mathrm{GKBO}} [t_1, \dots, t_m]$

  ensure monotonicity

Theorem [Middeldorp & Zantema '97]

$\succ_{\mathrm{GKBO}}$ is a simplification order, if

$[\![\cdot]\!]$ is weakly monotone and *strictly simple*: $[\![f(\dots, x, \dots) > x]\!]$

ensures subterm property

We know another way to ensure this!

WPO := GKBO + LPO

- **Definition**:

$s = f(s_1, \dots, s_n) \succ_{\text{WPO}} t$ iff

1. $[\![s > t]\!]$ or
2. $[\![s \geq t]\!]$ and

compare weights, from GKBO

a. $\exists i. s_i \succsim_{\text{WPO}} t$ or

b. $\forall j. s \succ_{\text{WPO}} t_j$ for $t = g(t_1, \dots, t_n)$ and

ensure subterm property, from LPO

i. $f > g$ or

ii. $f \sim g$ and $[s_1, \dots, s_n] \succ^{\text{lex}}_{\text{WPO}} [\text{t}_1, \dots, \text{t}_\text{m}]$

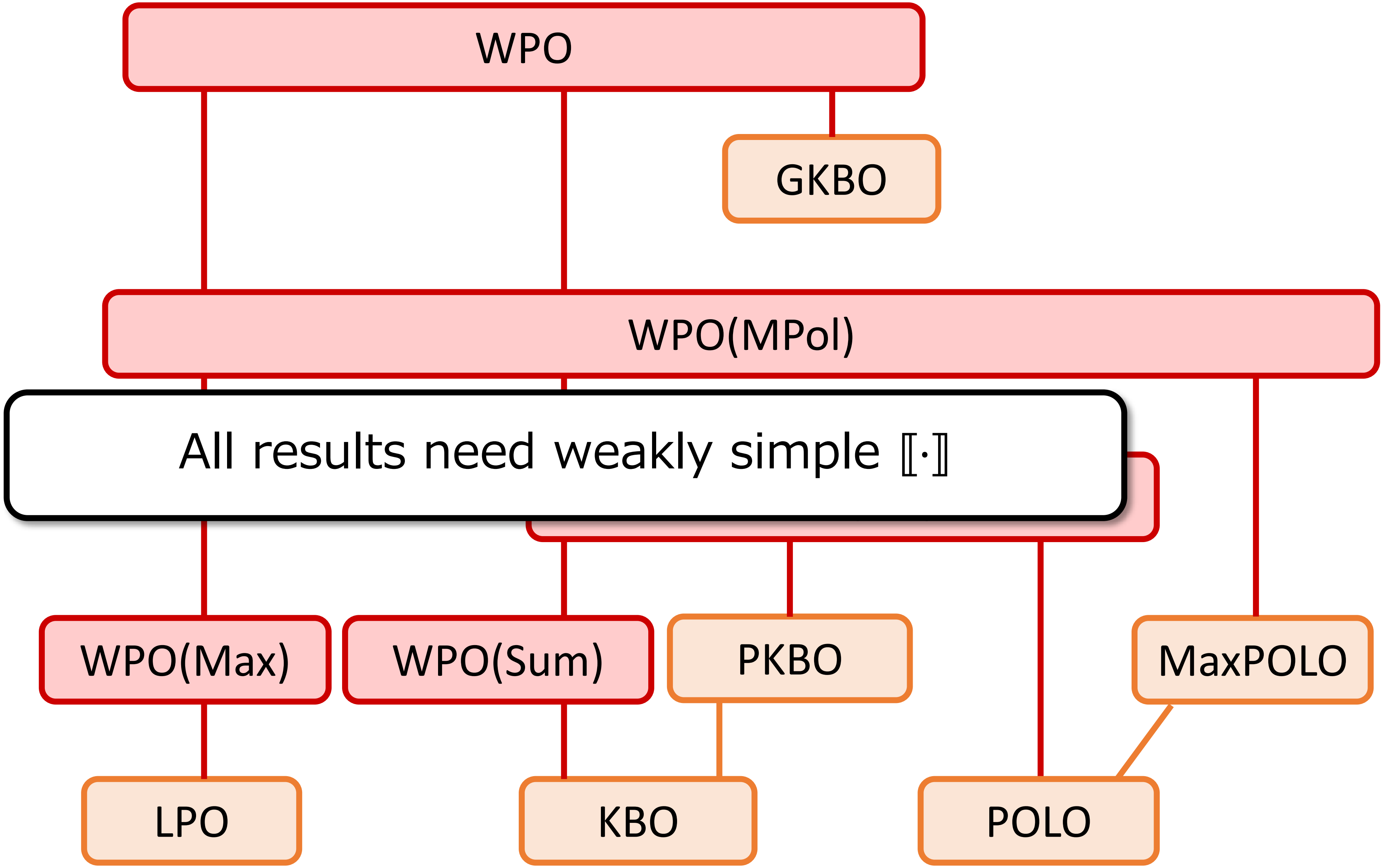
as in GKBO and LPO

Theorem: [Yamada+ PPDP'13]

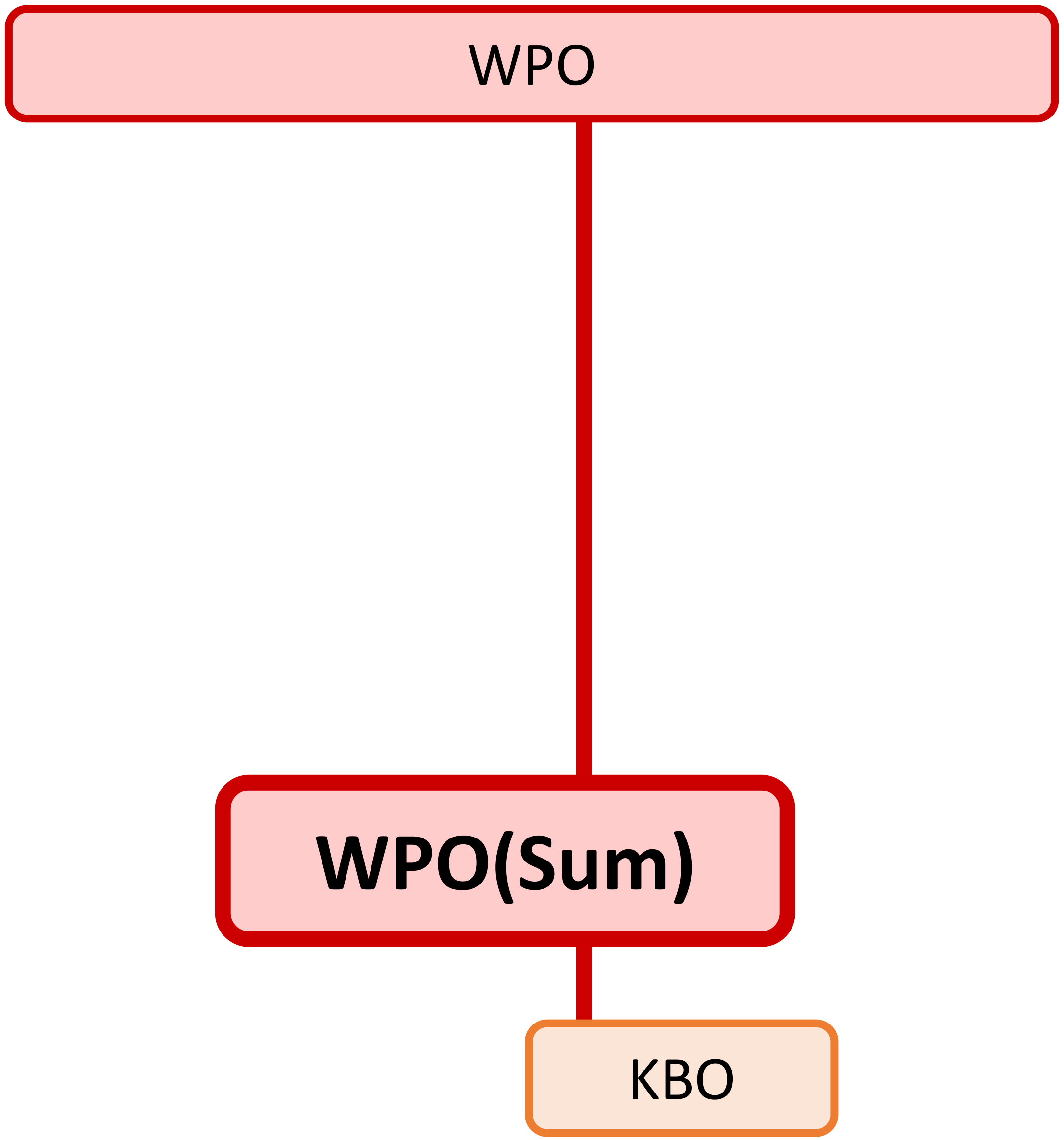
If $[\![\cdot]\!]$ is weakly monotone and *weakly simple*: $[\![f(\dots, x, \dots) \geq x]\!]$

then $\succ_{\text{WPO}}$ is a simplification order.

Map


WPO
GKBO
WPO(MPol)
All results need weakly simple ⟦·⟧
WPO(Max)
WPO(Sum)
PKBO
MaxPOLO
LPO
KBO
POLO

Map



WPO(Sum)

- **Definition**:

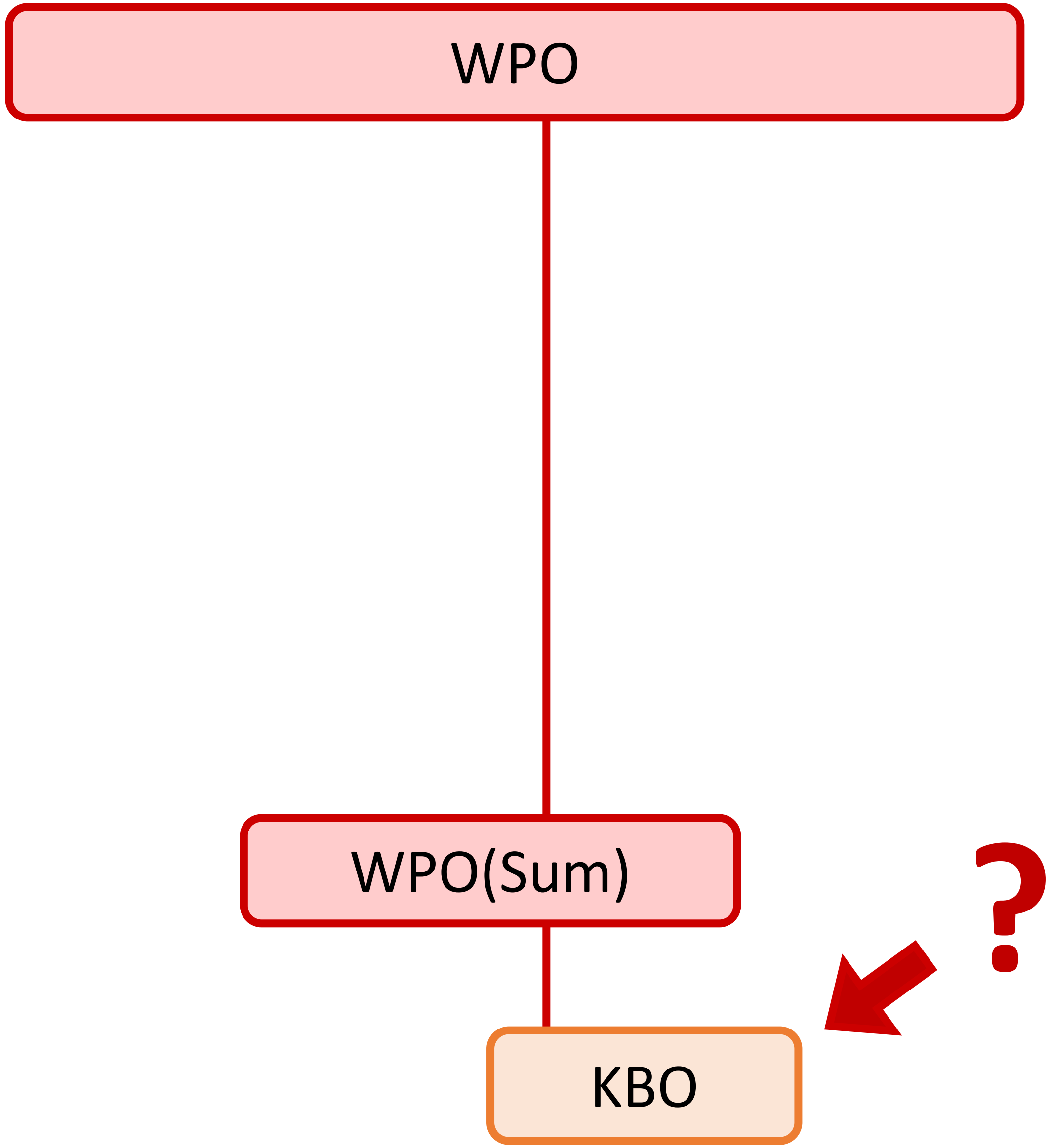
  The weakly monotone algebra $[\![\cdot]\!]_{sum}$ is defined by

  - Carrier: $(\mathbb{N}, >)$
  - Interpretation: $[\![f]\!]_{sum}(x_1, \ldots, x_n) = w_f + \sum_{i=1}^{n} x_i$

Corollary

$[\![\cdot]\!]_{sum}$ is weakly monotone and weakly simple.
Hence, $\succ_{\mathrm{WPO}(sum)}$ is a *simplification order*.

Map



KBO [Knuth & Bendix '70]

- **Definition** (in our notation):

  $s = f(s_1, \dots, s_n) \succ_{\text{KBO}} t$ iff

  1. $[\![s > t]\!]$ or

  2. $[\![s \geq t]\!]$ and

  a. $s = f^k(t)$ with $k > 0$, or

  b. $t = g(t_1, \dots, t_n)$ and

  i. $f \succ g$ or

  ii. $f \sim g$ and $[s_1, \dots, s_n] \succ_{\text{KBO}}^{lex} [t_1, \dots, t_m]$

a little simpler than WPO...

more complex

Theorem: $\succ_{\text{KBO}}$ is a simplification order if

A. Every $c \in \mathcal{F}_0$ has $w_c > 0$

B. Every $f \in \mathcal{F}_1$ with $w_f = 0$ is greatest in $\succ$ (*admissibility*)

WPO(Sum) ⊇ KBO

Proposition: WPO subsumes KBO.

Proof: If

A. Every $c \in \mathcal{F}_0$ has $\boldsymbol{w_c} > 0$

B. Every $f \in \mathcal{F}_1$ with $\boldsymbol{w_f} = 0$ is greatest in $\succ$

Then $\succ_{\mathrm{WPO}} = \succ_{\mathrm{KBO}}$

In other words, WPO(Sum) extends KBO by allowing

A. 0-weighted constants

B. 0-weighted unary symbols of any precedence

WPO(Sum) $\supsetneq$ KBO

- **Example A** (weight-0 constant):

$$f(a, \underline{b}) \to f(\underline{b}, f(\underline{b}, a))$$

$w(b) = \mathbf{0}$

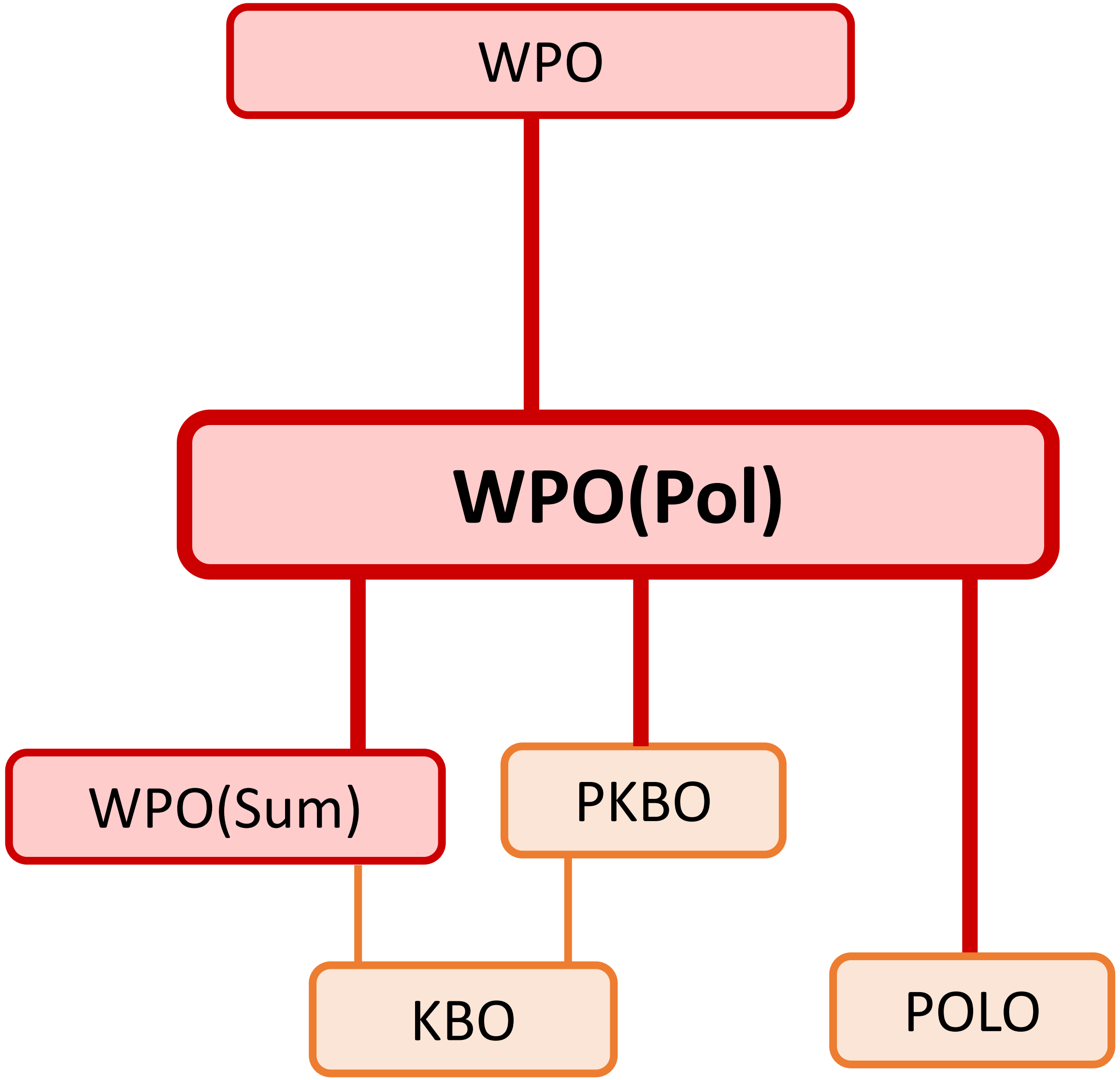
- **Example B** (non-admissibility):

$$\underline{f}(\underline{g}(x)) \to \underline{g}(\underline{g}(f(x)))$$

$w(g) = \mathbf{0}$

$f \succ g$

Map



WPO(Pol)

- **Definition**:

  Weakly monotone algebra $[\![\cdot]\!]_{pol}$:

  $[\![f]\!]_{pol}(x_1, \ldots, x_n)$: a *monotone polynomial* (coefficients ≥ 1)

Corollary

$\succ_{\mathrm{WPO}(pol)}$ is a *simplification order*.

Proof:

Because $[\![\cdot]\!]_{pol}$ is weakly simple (cf. [Zantema '01]).

WPO(Pol) ⊃ POLO ∪ PKBO

Proposition

WPO subsumes POLO.

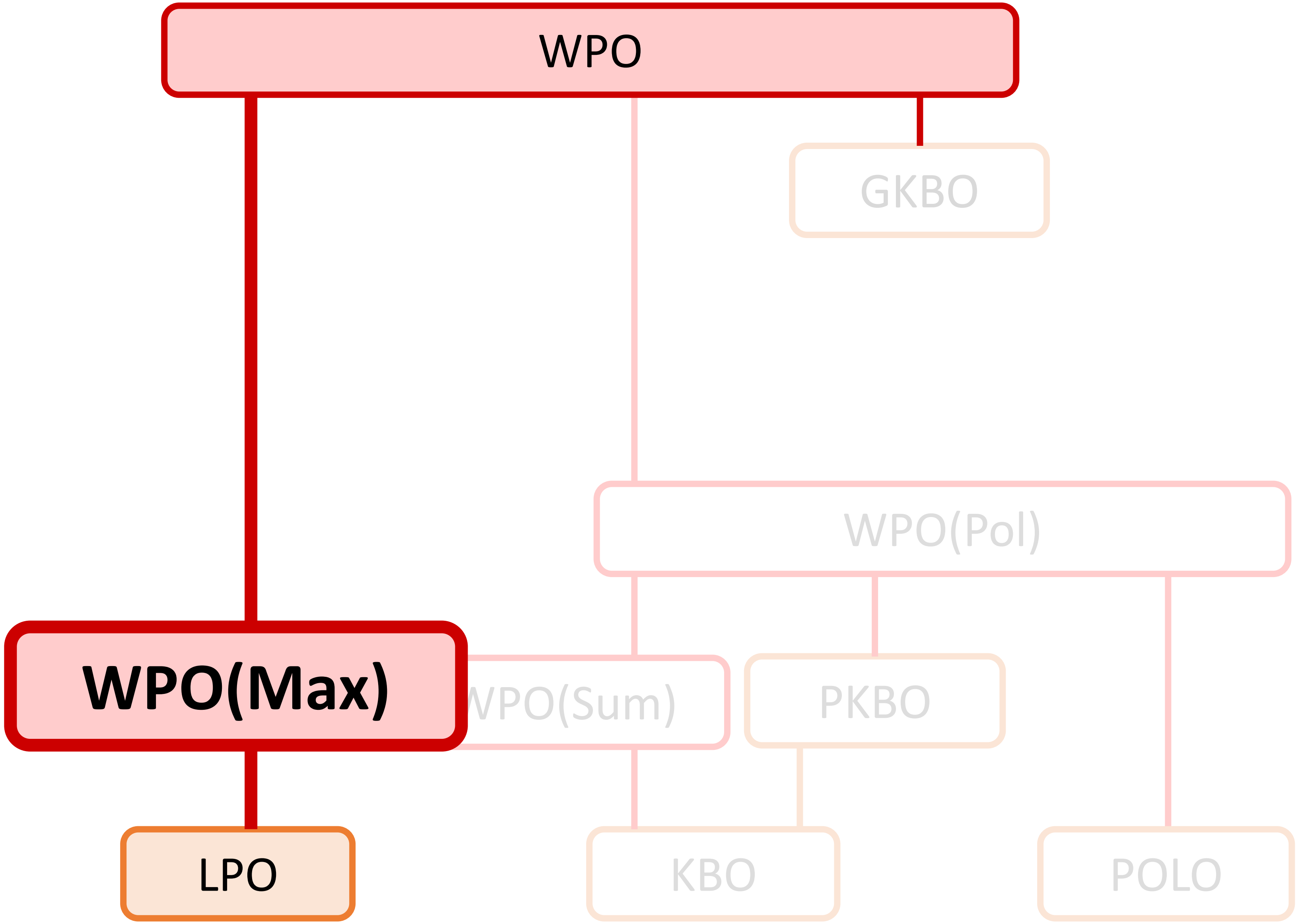
Proof: By definition, $[\![s > t]\!]$ implies $s \succ_{\mathrm{WPO}} t$ holds.

Corollary

WPO subsumes Polynomial KBO [Lankford '79].

Proof: Analogous to the KBO case.

Map



WPO(Max)

- **Definition**:

  The weakly monotone algebra $[\![\cdot]\!]_{max}$ is defined by

  - Carrier: $(\mathbb{N}, >)$
  - Interpretation: $[\![f]\!]_{max}(x_1, \dots, x_n) = \max_{i=1}^{n}\left(p_{f,i} + x_i\right)$

Corollary

$[\![\cdot]\!]_{max}$ is weakly monotone and weakly simple.

Hence, $>_{\mathrm{WPO}(max)}$ is a simplification order.

WPO(Max) $\supset$ LPO

Theorem

WPO subsumes LPO.

Proof

If $[\![f]\!](x_1, \ldots, x_n) = \mathbf{max}(x_1 + \mathbf{0}, \ldots, x_n + \mathbf{0})$ for any f,
then we have $\succ_{\mathrm{WPO}} = \succ_{\mathrm{LPO}}$.

- In other words,
  - WPO(max) extends LPO by giving $\boldsymbol{p}_{f,i} > \mathbf{0}$

WPO(Max) $\supsetneq$ LPO

- **Example**:

$$\mathcal{R}_3 := \begin{cases} f(x,y) \to g(x) \\ f(g(x),y) \to f(x,g(x)) \\ f(x,g(y)) \to f(y,y) \end{cases}$$

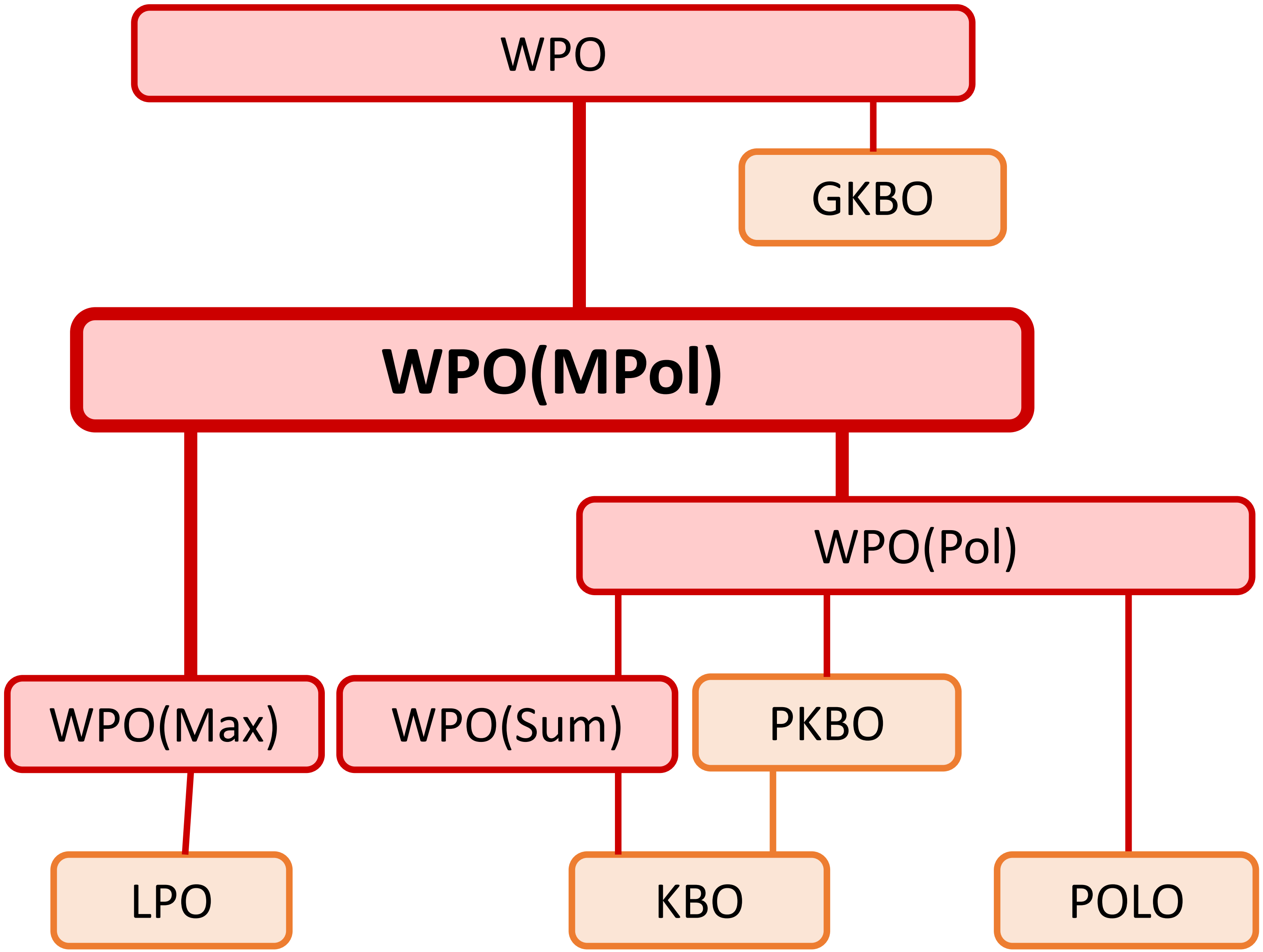
$f \succ g$

lexicographically

$[\![f]\!](x,y) = max(x,y)$

$[\![g]\!](y) = y + 1$

Map



WPO(MPol) ⊃ many

- **Definition**:

  The *weakly monotone algebra* $[\![\cdot]\!]_{mpol}$ induced by a *weight status* $\mathrm{ws} : \mathcal{F} \to \{\mathrm{pol}, \mathrm{max}\}$:

$$[\![f]\!]_{mpol}(\dots) = \begin{cases} [\![f]\!]_{pol}(\dots) & \text{if } ws(f) = \mathrm{pol} \\ [\![f]\!]_{max}(\dots) & \text{if } ws(f) = \mathrm{max} \end{cases}$$

Corollary

$\succ_{WPO(mpol)}$ is a *simplification order*.

Theorem

WPO subsumes KBO, LPO, POLO, PKBO.

WPO $\subseteq$ SPO

- **Definition** [Kamin&Levy'80]:

  $s = f(s_1, \dots, s_n) \succ_{\mathrm{SPO}} t$ iff

  a. $\exists i. s_i \gtrsim_{\mathrm{SPO}} t$ or

  b. $\forall j. s \succ_{\mathrm{SPO}} t_j$ for $t = g(t_1, \dots, t_n)$ and

  1. $[\![s > t]\!]$ or

  2. $[\![s \geq t]\!]$ and $[s_1, \dots, s_n] \succ_{\mathrm{SPO}}^{\mathrm{lex}} [\mathrm{t}_1, \dots, \mathrm{t}_\mathrm{m}]$

- Take $>$ as $\langle >, \succ \rangle_{\mathrm{lex}}$ … but WPO is monotone

WSPO?

- **Definition**:

  $s = f(s_1, \dots, s_n) \succ_{\mathrm{WSPO}} t$ iff

  1. $[\![s > t]\!]_1$ or
  2. $[\![s \geq t]\!]_1$ and

     a. $\exists i. s_i \succsim_{\mathrm{WSPO}} t$ or

     b. $\forall j. s \succ_{\mathrm{WSPO}} t_j$ for $t = g(t_1, \dots, t_n)$ and

        i. $[\![s > t]\!]_2$ or

        ii. $[\![s \geq t]\!]_2$ and $[s_1, \dots, s_n] \succ_{\mathrm{WPO}}^{\mathrm{lex}} [\mathrm{t}_1, \dots, \mathrm{t}_\mathrm{m}]$

as in SPO

<u>**Theorem**</u>: (?)

If $[\![\cdot]\!]_1$ and $[\![\cdot]\!]_2$ are weakly monotone and $[\![\cdot]\!]_1$ is weakly simple, then $\succ_{\mathrm{WPO}}$ is a simplification order.

Implementation: SMT encoding

- **Definition** (WPO, again):

  $s = f(s_1, \dots, s_n) \succ_{\mathrm{WPO}} t$ iff

  1. $[\![s > t]\!]$ or
  2. $[\![s \geq t]\!]$ and

     a. $\exists i. s_i \succsim_{\mathrm{WPO}} t$ or

     b. $\forall j. s \succ_{\mathrm{WPO}} t_j$ for $t = g(t_1, \dots, t_n)$ and

        i. $f \succ g$ or

        ii. $f \sim g$ and $[s_1, \dots, s_n] \succ_{\mathrm{WPO}}^{\mathrm{lex}} [\mathrm{t}_1, \dots, \mathrm{t}_\mathrm{m}]$

Implementation: SMT encoding

- **Definition** (WPO, again):

$s = f(s_1, \ldots, s_n) \succ_{\mathrm{WPO}} t$:=

1. $[\![s > t]\!]$ ∨
2. $[\![s \geq t]\!]$ ∧

a. $s_1 \gtrsim_{\mathrm{WPO}} t \vee \cdots \vee s_n \gtrsim_{\mathrm{WPO}} t \vee$

b. $s \succ_{\mathrm{WPO}} t_1 \wedge \cdots \wedge s \succ_{\mathrm{WPO}} t_m \wedge$ (when $t = g(t_1, \ldots, t_n)$)

i. $p_f > p_g \vee$

ii. $p_f = p_g \wedge [s_1, \ldots, s_n] \succ_{\mathrm{WPO}}^{\mathrm{lex}} [\mathrm{t}_1, \ldots, \mathrm{t}_\mathrm{m}]$

Experimental Comparison

order	algebra	non-duplicating (439)			duplicating (1024)		
		yes	T.O.	time (s)	yes	T.O.	time (s)
POLO	$\mathcal{S}um$	41	0	3.54	-	-	-
LPO	-	90	0	20.94	90	0	22.64
KBO	-	115	0	4.51	-	-	-
WPO	$\mathcal{S}um$	**135**	0	31.47	-	-	-
WPO	$\mathcal{MS}um$	**135**	0	31.55	**138**	0	40.06
WPO	$\mathcal{M}ax$	116	0	59.96	125	0	35.23
POLO	$\mathcal{P}ol$	81	3	190.45	12	9	812.18
PKBO	$\mathcal{P}ol$	125	0	10.38	22	9	1146.65
MAT	-	136	3	246.73	20	15	1638.92
WPO	$\mathcal{MP}ol$	**147**	0	276.05	**138**	9	982.71

Contents

- abstract things
- monotone WPO (weighted path order)
- **weakly monotone WPO**
- constrained WPO
- non-monotone WPO

Dependency Pairs [Arts & Giesl '00]

- Consider TRS:

$$R_{\mathrm{sum}} = \begin{cases} \mathrm{sum}(\mathrm{nil}) \to 0 \\ \boxed{\mathrm{sum}(x :: xs)} \to x + \boxed{\mathrm{sum}(xs)} \end{cases}$$

  - There is one dependency:
    $\mathcal{P} = \{ \boxed{\mathrm{sum}(x :: xs)} \to \boxed{\mathrm{sum}(xs)} \}$

  - Then we want a *weakly monotone* $[\![\cdot]\!]$ s.t.

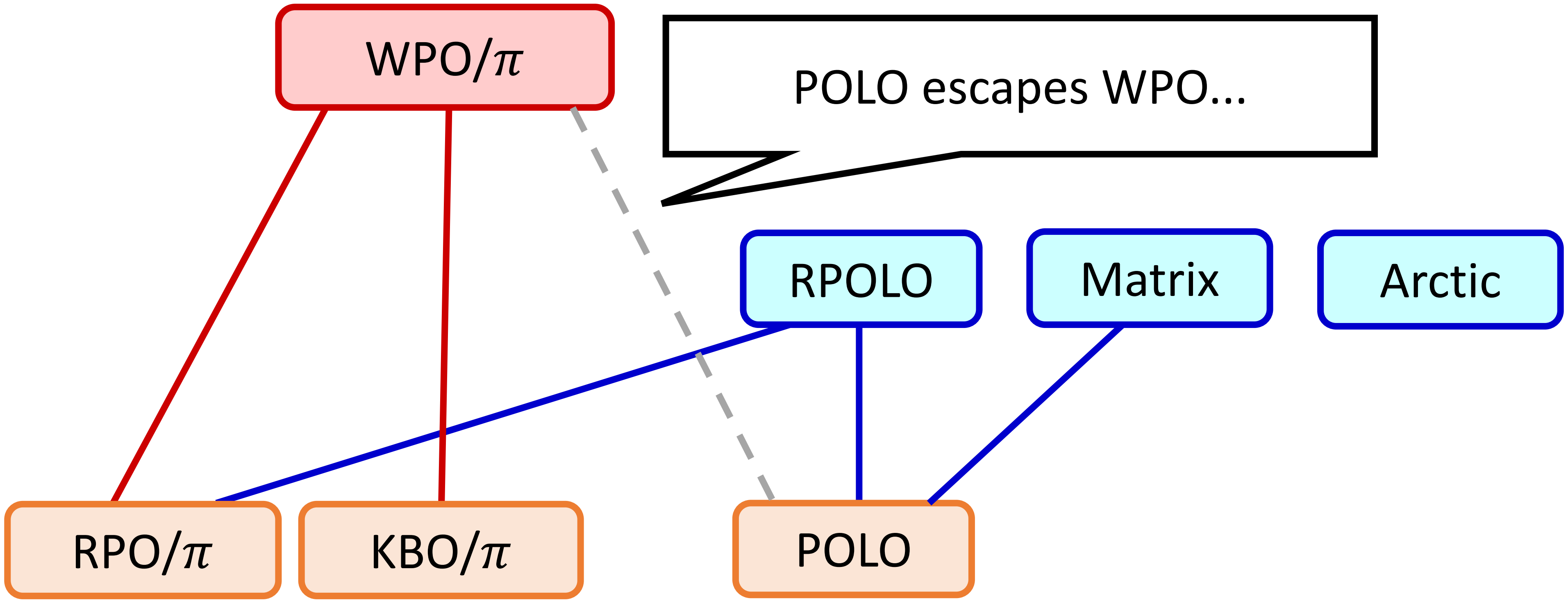
$$[\![\mathrm{sum}(\mathrm{nil}) \geq 0]\!]$$
$$[\![\mathrm{sum}(x :: xs) \geq x + \mathrm{sum}(xs)]\!]$$
$$[\![\mathrm{sum}^{\#}(x :: xs) > \mathrm{sum}^{\#}(xs)]\!]$$

How to design such $[\![\cdot]\!]$?

Existing weakly monotone methods

- **Polynomial interpretation** (with 0-coefficients) [Arts & Giesl '00]
- Monotone method + **argument filtering** π
  - **KBO**/π [Arts & Giesl '00]
  - **RPO**/π [Arts & Giesl '00]
  - **WPO**/π [Yamada+ '13]
- **Matrix** interpretations [Endrullis+ '08]
- **Arctic** interpretations [Koprowski & Waldmann '09]
- **RPOLO**: RPO + Polynomial interpretation [Bofill+ '12]

Map: weakly monotone methods (2013)



WPO $\not\supseteq$ POLO

- Counterexample:

$$\text{If} \begin{cases} [\![\mathrm{f}]\!](x) = 2x \\ [\![\mathrm{a}]\!] = 0 \\ [\![\mathrm{b}]\!] = 1 \end{cases} \text{ then } \begin{cases} [\![\mathrm{a} \geq \mathrm{f(a)}]\!] \\ [\![\mathrm{f(b)} > \mathrm{b}]\!] \end{cases} \text{ but } \begin{cases} \mathrm{a} \prec_{\mathrm{WPO}} \mathrm{f(a)} \\ \mathrm{f(b)} \succ_{\mathrm{WPO}} \mathrm{b} \end{cases}$$

- Idea for rescue:

ignore argument of f in LPO-style recursive check, but count its weight

not possible by argument filtering

Weakly monotone WPO

- **Definition**:

  $s = f(s_1, \dots, s_n) {}_{(}\gtrsim_{)\mathrm{WPO}} t$ iff

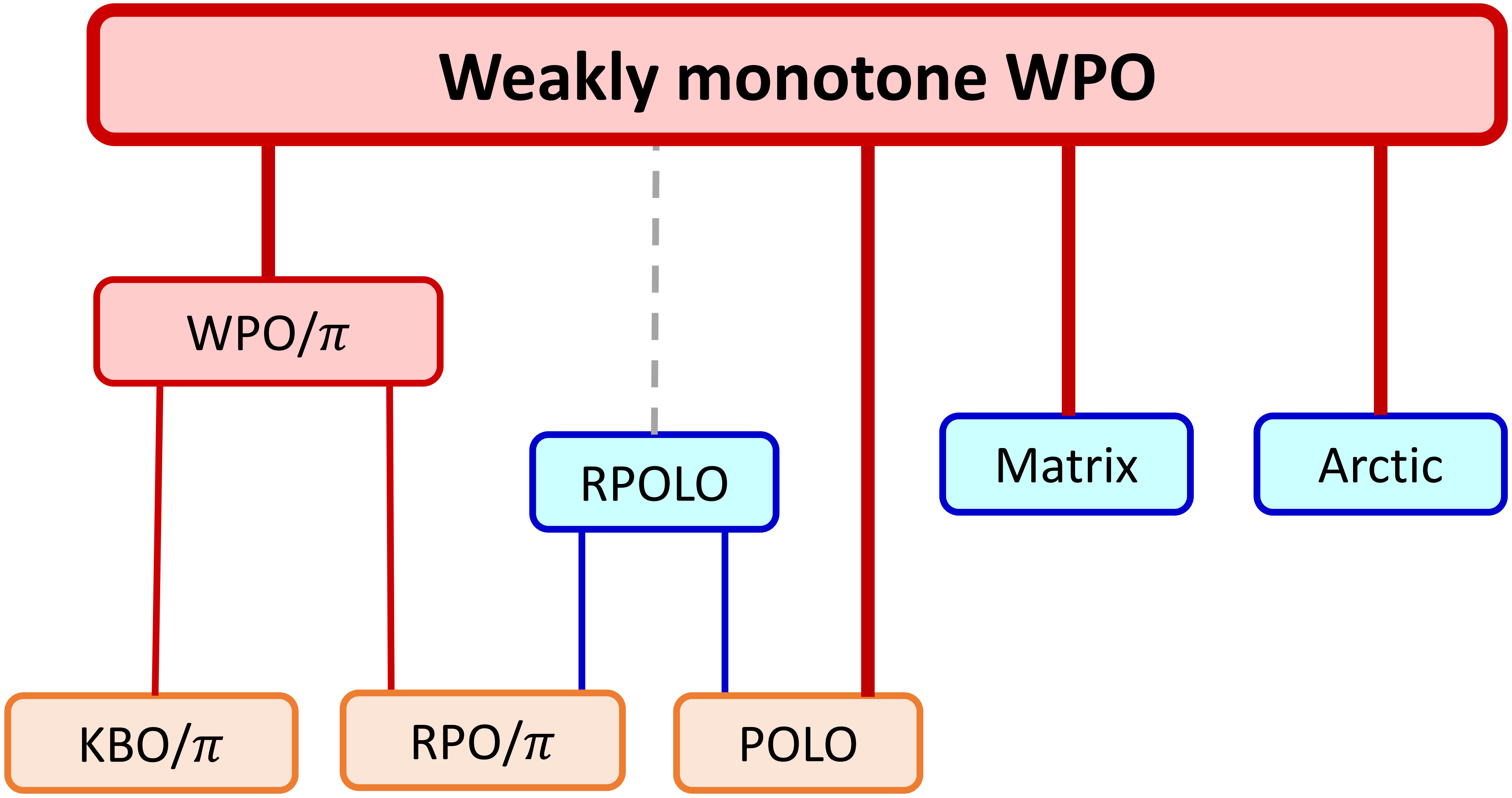
  1. $[\![s > t]\!]$ or
  2. $[\![s \geq t]\!]$ and
     a. $\exists i \in \boldsymbol{\pi}(f).\, s_i \gtrsim_{\mathrm{WPO}} t$ or
     b. $\forall j \in \boldsymbol{\pi}(g).\, s \succ_{\mathrm{WPO}} t_j$ for $t = g(t_1, \dots, t_n)$ and
        i. $f \succ g$ or
        ii. $f \sim g$ and $\boldsymbol{\pi}(f)[s_1, \dots, s_n] {}_{(}\gtrsim_{)\mathrm{WPO}}^{\mathrm{lex}} \boldsymbol{\pi}(g)[t_1, \dots, t_m]$

Theorem [Yamada+ '14]

If $[\![\cdot]\!]$ is weakly monotone and $\boldsymbol{\pi}$*-simple* : $\forall i \in \boldsymbol{\pi}(f).\, [\![f(\dots, x_i, \dots) \geq x_i]\!]$,

then WPO is well-founded and weakly monotone.

Map: Weakly monotone methods (2014)



Experimental Comparison

reduction pair	algebra	argument filter			partial status		
		yes	T.O.	time (s)	yes	T.O.	time (s)
POLO	$\mathcal{S}um$	512	0	111.51	-	-	-
LPO	-	502	0	361.41	-	-	-
KBO	-	497	3	900.79	520	4	1109.59
WPO	$\mathcal{S}um$	514	2	830.21	560	4	1105.42
WPO	$\boldsymbol{\mathcal{MS}um}$	578	6	1289.17	**675**	12	1755.66
WPO	$\mathcal{M}ax$	548	6	1230.17	637	13	1855.59
POLO	$\mathcal{P}ol$	544	19	1958.44	-	-	-
PKBO	$\mathcal{P}ol$	516	187	15665.26	539	178	11579.28
MAT	-	**645**	480	32367.26	-	-	-
WPO	$\mathcal{MP}ol$	560	88	7678.43	672	94	9269.36

Contents

- abstract things
- monotone WPO (weighted path order)
- weakly monotone WPO
- **constrained WPO (new)**
- non-monotone WPO

Constrained TRSs [Furuichi+ '08, Kop&Nishida '13]

- **Example**:

$$\mathcal{R} = \left\{ \begin{array}{ll} \mathrm{fact}(x) \to 1 & [x \leq 0] \\ \mathrm{fact}(x) \to x * \mathrm{fact}(x-1) & [x > 0] \end{array} \right\}$$

- Component 1: ***base logic*** $[\![\cdot]\!]_{\mathcal{B}}$
  - **computation step**: $s \to_{\mathcal{S}} t \implies [\![s]\!]_{\mathcal{B}} = [\![t]\!]_{\mathcal{B}}$
- Component 2: ***constrained rules*** $\mathcal{R}$: $l \to r\ [\varphi]$
  - define "rule step": $l\theta \xrightarrow[\langle l,\phi,r \rangle]{\epsilon} r\theta \Leftarrow [\![\varphi\theta]\!]_{\mathcal{B}} = \mathrm{True}$

Constrained dependency pairs [Kop '13]

- **Example**:

$$\mathcal{R} = \left\{ \begin{array}{ll} \mathrm{fact}(x) \to 1 & [x \leq 0] \\ \mathrm{fact}(x) \to x * \mathrm{fact}(x-1) & [x > 0] \end{array} \right\}$$

  - $\mathrm{DP}(\mathcal{R}) = \mathcal{P} = \{ \mathrm{fact}(x) \to \mathrm{fact}(x-1)\ [x > 0] \}$
- So we want a weakly monotone $[\![\cdot]\!]$ s.t.
  - $\mathcal{R} \subseteq [\geq] = \left\{ \rho \mid s \to_\rho t \ \subseteq\ [\![s]\!] \geq [\![t]\!] \right\}$
  - $\mathcal{S} \subseteq \geq$
  - $\mathcal{P} \subseteq [>]$

Constrained WPO

- **Definition**:

$s = f(s_1, \ldots, s_n) \,_{(}\!\succsim_{)}\, _{\mathrm{WPO}[\varphi]}\; t$ iff

1. $[\![\varphi \Rightarrow s > t]\!]$ or
2. $[\![\varphi \Rightarrow s \geq t]\!]$ and
   a. $\exists i \in \pi(f).\, s_i \succsim_{\mathrm{WPO}[\varphi]} t$ or
   b. $\forall j \in \pi(g).\, s \succ_{\mathrm{WPO}[\varphi]} t_j$ for $t = g(t_1, \ldots, t_n)$ and
      i. $f \succ g$ or
      ii. $f \sim g$ and $\pi(f)[s_1, \ldots, s_n] \,_{(}\!\succsim_{)}\,^{\mathrm{lex}}_{\mathrm{WPO}[\varphi]}\, \pi(g)[t_1, \ldots, t_m]$

Lemma [new]:

If $[\![\cdot]\!]$ respects $[\![\cdot]\!]_{\mathcal{B}}$, then $l \succ_{\mathrm{WPO}[\varphi]} r$ implies $\xrightarrow[l \to r[\varphi]]{\epsilon} \subseteq \succ_{\mathrm{WPO}}$

$[\![s]\!] = [\![s]\!]_{\mathcal{B}}$ if $s \in T(\mathcal{B}, \emptyset)$

Proof

Lemma

If $[\![\cdot]\!]$ respects $[\![\cdot]\!]_{\mathcal{B}}$, then $l \succ_{\mathrm{WPO}[\varphi]} r$ implies $\xrightarrow[l \to r[\varphi]]{\epsilon} \subseteq \succ_{\mathrm{WPO}}$

Proof. Induction on $|l| + |r|$.

Show $l\theta \succ_{\mathrm{WPO}} r\theta$ if $[\![\varphi\theta]\!]_{\mathcal{B}} = \mathrm{True}$, so $[\![\varphi\theta]\!] = \mathrm{True}$

Case 1: Suppose $[\![\varphi \Rightarrow l > r]\!]$. Then $[\![\varphi\theta \Rightarrow l\theta > r\theta]\!]$ so $[\![l\theta > r\theta]\!]$, so $l\theta \succ_{\mathrm{WPO}} r\theta$. by case 1 (unconstrained WPO).

Proof

Lemma

If $[\![\cdot]\!]$ respects $[\![\cdot]\!]_{\mathcal{B}}$, then $l \succ_{\text{WPO}[\varphi]} r$ implies $\xrightarrow[l \to r[\varphi]]{\epsilon} \subseteq \gtrsim_{\text{WPO}}$

Proof. Induction on $|l| + |r|$.

Show $l\theta \succ_{\text{WPO}} r\theta$ if $[\![\varphi\theta]\!]_{\mathcal{B}} = \text{True}$, so $[\![\varphi\theta]\!] = \text{True}$

Case 1: ...

Case 2: Suppose $[\![\varphi \Rightarrow l \geq r]\!]$. Then $[\![\varphi\theta \Rightarrow l\theta \geq r\theta]\!]$ so $[\![l\theta \geq r\theta]\!]$.

Case a: If $\exists i \in \pi(f).\, l_i \gtrsim_{\text{WPO}[\varphi]} r$ then by IH $l_i\theta \gtrsim_{\text{WPO}} r\theta$,

so $l\theta \succ_{\text{WPO}} r\theta$ by case 2a.

Proof

Lemma

If $[\![\cdot]\!]$ respects $[\![\cdot]\!]_{\mathcal{B}}$, then $l \succ_{\mathrm{WPO}[\varphi]} r$ implies $\xrightarrow[l \to r[\varphi]]{\epsilon} \subseteq \gtrsim_{\mathrm{WPO}}$

Proof. Induction on $|l| + |r|$.

Show $l\theta \succ_{\mathrm{WPO}} r\theta$ if $[\![\varphi\theta]\!]_{\mathcal{B}} = \mathrm{True}$, so $[\![\varphi\theta]\!] = \mathrm{True}$

Case 1: ...

Case 2: Suppose $[\![\varphi \Rightarrow l \geq r]\!]$. Then $[\![\varphi\theta \Rightarrow l\theta \geq r\theta]\!]$ so $[\![l\theta \geq r\theta]\!]$.

Case a: ...

Case b: If $\forall j \in \pi(g).\, l \succ_{\mathrm{WPO}[\varphi]} r_j$ then by IH $l\theta \succ_{\mathrm{WPO}} r_j\theta$.

Case i: If $f \succ g$ then $l\theta \succ_{\mathrm{WPO}} r\theta$.

Case ii: If $f \sim g$, lexicographic. From IH. ◻

Are we happy with it?

- We've got weakly monotone constrained WPO

  if $[\![\cdot]\!]$ is weakly monotone, and
  $[\![\cdot]\!]$ respects $[\![\cdot]\!]_{\mathcal{B}}$ ...

Base logic has to be weakly monotone!

Contents

- abstract things
- monotone WPO (weighted path order)
- weakly monotone WPO
- constrained WPO
- **non-monotone WPO (new, rough)**

Non-monotone method

- Monotonicity specification: $\boldsymbol{\mu}(f, i) \in \{0, +1, -1, \pm 1\}$
- $[\![\cdot]\!]$ is ***μ-monotone*** if
  - $\boldsymbol{\mu}(f, i) = +1$ implies $a_i \geq a_i' \Longrightarrow [\![f]\!](\dots a_i \dots) \geq [\![f]\!](\dots a_i' \dots)$ *monotone*
  - $\boldsymbol{\mu}(f, i) = -1$ implies $a_i \geq a_i' \Longrightarrow [\![f]\!](\dots a_i \dots) \leq [\![f]\!](\dots a_i' \dots)$ *antitone*
  - $\boldsymbol{\mu}(f, i) = 0$ implies $[\![f]\!](\dots a_i \dots) = [\![f]\!](\dots a_i' \dots)$ *constant*

Theorem [Giesl+ '07]

Innermost termination can be proved by well-founded **μ-monotone** algebras.

Non-monotone WPO

- **Definition** (again):

$$s = f(s_1, \dots, s_n) \,_{(}\!\gtrsim\!_{)}\, _{\mathrm{WPO}[\phi]}\; t \quad \text{iff}$$

1. $[\![\phi \Rightarrow s > t]\!]$ or
2. $[\![\phi \Rightarrow s \geq t]\!]$ and
   - a. $\exists i \in \pi(f).\, s_i \gtrsim_{\mathrm{WPO}[\phi]} t$ or
   - b. $\forall j \in \pi(g).\, s \succ_{\mathrm{WPO}[\phi]} t_j$ for $t = g(t_1, \dots, t_n)$ and
     - i. $f \succ g$ or
     - ii. $f \sim g$ and $\pi(f)[s_1, \dots, s_n] \,_{(}\!\gtrsim\!_{)}{}^{\mathrm{lex}}_{\mathrm{WPO}[\phi]}\; \pi(g)[t_1, \dots, t_m]$

Theorem [new]:

If $[\![\cdot]\!]$ is π-simple and ***μ*-monotone**, and $i \in \pi(f) \Longrightarrow$ $\boldsymbol{\mu}(f, i) = +1$

then WPO is well-founded and ***μ*-monotone**.

TODO...

- Write papers
- Lift "innermost"? Understand [Fuhs+'08]
- Implement
- Introduce TermComp category: Constrained TRS

Generalized usable rules

- **Definition**: U w.r.t. $\mathcal{R}$ and $\boldsymbol{\mu}$:
  - $U(x) = \emptyset$
  - $U\big(f(s_1, \dots, s_n)\big) \supseteq \{\langle l, r\rangle\} \cup U(r)$ for every $l \to r \in \mathcal{R}$ and $\mathrm{root}(l) = f$ and $\supseteq U(s_1)^{\boldsymbol{\mu(f,1)}} \cup \cdots \cup U(s_n)^{\boldsymbol{\mu(f,n)}}$
- Extend μ to terms and positions:
  - $\mu(s, \epsilon) = +1$
  - $\mu(f(s_1, \dots, s_n), ip) = \mu(f, i) \cdot \mu(s_i, p)$

$\sqsupset^{+1} = \sqsupset$	$\sqsupset^{-1} = \sqsubset$
$\sqsupset^{0} = \emptyset$	$\sqsupset^{\pm 1} = \sqsupset \cup \sqsubset$

Conjecture:

Suppose whenever $l \to r \in \mathcal{P} \cup \mathcal{R}$ and $r|_p = x \in V$,

either $\mu(r, p) = 0$ or $\mu(r, p) = +1$ and $\exists q.\ l|_q = x \wedge \mu(l, q) = +1$

$\mathcal{P} \cup U \subseteq [\geq]$. Then $\langle \mathcal{P}, \mathcal{R}\rangle$ is terminating if $\langle \mathcal{P} \setminus [>], \mathcal{R}\rangle$ is.