

Termination of $\lambda\Pi$ -Calculus Modulo Rewriting using Size-Change Principle

Work in progress

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école —
normale —
supérieure —
paris-saclay —



- 1 $\lambda\Pi$ -Calculus Modulo Rewriting
- 2 Size-Change Termination
- 3 Computability Closure
- 4 The Extended Criterion

Typing rule for application :

$$\frac{\Gamma \vdash t : \forall(x : A).B \quad \Gamma \vdash u : A}{\Gamma \vdash tu : B [u/x]}$$

`N : TYPE`

`0 : N`

`s : N ⇒ N`

`A : TYPE`

`List : N ⇒ TYPE`

`nil : List 0`

`cons : ∀(n:N).A ⇒ List n ⇒ List (s n)`

Conversion :

$$\frac{\Gamma \vdash t : A \quad A \longleftrightarrow^* B \quad \Gamma \vdash B : \text{TYPE}}{\Gamma \vdash t : B}$$

$\mathbb{N} : \text{TYPE}$

$0 : \mathbb{N}$

$s : \mathbb{N} \Rightarrow \mathbb{N}$

$+ : \mathbb{N} \Rightarrow \mathbb{N} \Rightarrow \mathbb{N}$

$0 + n \longrightarrow n$

$(s\ m) + n \longrightarrow s\ (m + n)$

$(m + n) + p \longrightarrow m + (n + p)$

Higher-order rewrite rules

`map`: $(A \Rightarrow A) \Rightarrow \forall(n:\mathbb{N}). \text{List } n \Rightarrow \text{List } n$

`map` `f` `0` `nil` \longrightarrow `nil`

`map` `f` `_` (`cons` `n` `x` `l`) \longrightarrow `cons` `n` (`f` `x`) (`map` `f` `n` `l`)

Type-level rewrite rules

`NArrows` : $\mathbb{N} \Rightarrow \text{TYPE}$

`NArrows` `0` \longrightarrow `N`

`NArrows` (`s` `n`) \longrightarrow `N` \Rightarrow (`NArrows` `n`)

We want to prove **strong normalization** of $\longrightarrow_{\beta} \cup \longrightarrow_{\mathcal{R}}$ for *typable* terms in the $\lambda\Pi$ -calculus modulo $\longleftrightarrow_{\beta\mathcal{R}}^*$

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For \mathcal{R} orthogonal

- 1) No infinite DP-chains \Rightarrow Weak normalization of typable terms,
- 2) Size-Change Termination \Rightarrow No infinite DP-chains.

For \mathcal{R} orthogonal **confluent**

- 1) No infinite DP-chains \Rightarrow ~~Weak~~ **Strong** normalization of typable terms,
- 2) Size-Change Termination \Rightarrow No infinite DP-chains.

Definition (Dependency pair)

$(f \bar{p}, g \bar{u})$ is a dependency pair if $f \bar{p} \longrightarrow C[g \bar{u}]$ is a rewrite rule.

Definition (Instanced call)

$f \bar{t} \widetilde{>} g \bar{v}$ if:

- $(f \bar{p}, g \bar{u})$ is a dependency pair,
- $\bar{t} \longrightarrow^* \bar{p}\sigma$,
- $\bar{u}\sigma = \bar{v}$.

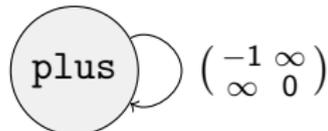
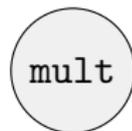
$\widetilde{>}$ is well-founded = no infinite DP-chains

mult

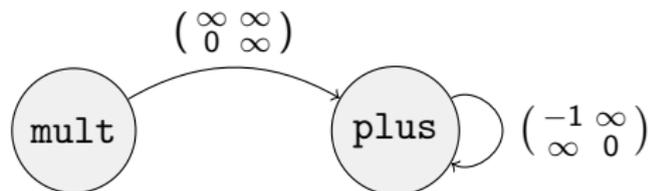
plus

$$(s \ m) + n \longrightarrow s \ (m + n)$$

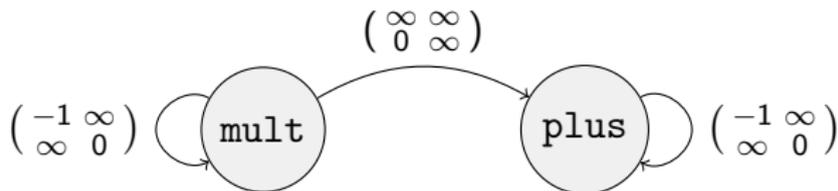
$$(s \ m) \times n \longrightarrow n + (m \times n)$$



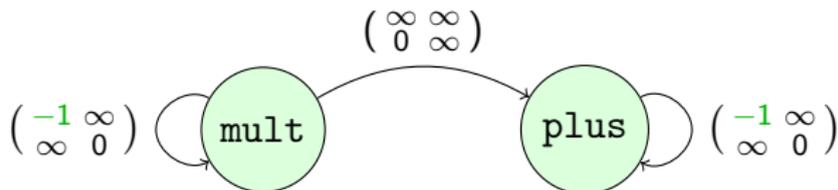
$(s\ m) + n \rightarrow s\ (m + n)$	$\begin{matrix} & m & n \\ s\ m & \begin{pmatrix} -1 & \infty \\ \infty & 0 \end{pmatrix} \\ n & \end{matrix}$
$(s\ m) \times n \rightarrow n + (m \times n)$	



$(s \ m) + n \rightarrow s \ (m + n)$	$\begin{matrix} m & n \\ sm & \begin{pmatrix} -1 & \infty \\ \infty & 0 \end{pmatrix} \\ n & \end{matrix}$
$(s \ m) \times n \rightarrow n + (m \times n)$	$\begin{matrix} n & m \times n \\ sm & \begin{pmatrix} \infty & \infty \\ 0 & \infty \end{pmatrix} \\ n & \end{matrix}$



$(s \ m) + n \rightarrow s \ (m + n)$	$s \ m \begin{pmatrix} m & n \\ -1 & \infty \\ \infty & 0 \end{pmatrix}$
$(s \ m) \times n \rightarrow n + (m \times n)$	$s \ m \begin{pmatrix} n & m \times n \\ \infty & \infty \\ 0 & \infty \end{pmatrix}$
	$s \ m \begin{pmatrix} m & n \\ -1 & \infty \\ \infty & 0 \end{pmatrix}$



$(s \ m) + n \rightarrow s \ (m + n)$	$s \ m \begin{pmatrix} m & n \\ -1 & \infty \\ \infty & 0 \end{pmatrix}$
$(s \ m) \times n \rightarrow n + (m \times n)$	$s \ m \begin{pmatrix} n & m \times n \\ \infty & \infty \\ 0 & \infty \end{pmatrix}$

Orthogonality is a restriction

Associativity and distributivity

$$(x + y) + z \longrightarrow x + (y + z)$$

$$(x + y) \times z \longrightarrow (x \times z) + (y \times z)$$

Signed integers

\mathbb{Z} : TYPE

0 : \mathbb{Z}

s : $\mathbb{Z} \Rightarrow \mathbb{Z}$

p : $\mathbb{Z} \Rightarrow \mathbb{Z}$

s (p x) \longrightarrow x

p (s x) \longrightarrow x

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Given a well-founded precedence \preceq .

Blanqui, 2005

Typable terms are SN, if for all $f \bar{I} \rightarrow r \in \mathcal{R}$, $r \in CC(f \bar{I})$

$CC(f \bar{I})$ is the set of typable terms, where function application is restricted as follows:

$$\frac{g \prec f \vee (g \simeq f \wedge \bar{m} <_{mul} \bar{I}) \quad \bar{m} \in CC(f \bar{I})}{g \bar{m} \in CC(f \bar{I})}$$

Property

$CC(f \bar{I}) \subseteq SN$ whenever \bar{I} are strongly normalizing.

It requires a strict decrease at each call.

Mutually recursive

```
filter : (A ⇒ Bool) ⇒ List ⇒ List
```

```
filter f (cons x l) → bis (f x) f x l
```

```
bis true  f x l → cons x (filter f l)
```

```
bis false f x l → filter f l
```

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Theorem

$\longrightarrow_{\beta} \cup \longrightarrow_{\mathcal{R}}$ is SN on typable term modulo $\longleftarrow_{\beta\mathcal{R}}^*$ if:

- $\longrightarrow_{\beta} \cup \longrightarrow_{\mathcal{R}}$ is confluent (if there are type-level rules only),
- $\longrightarrow_{\beta} \cup \longrightarrow_{\mathcal{R}}$ preserves typing,
- \mathcal{R} satisfies SCT,
- the right-hand side of every rewrite rule is typable.

The proof relies on an adaptation of Girard's reducibility candidates to rewriting [Girard, 1972] [Jouannaud, Okada, 1997] :

We interpret every type T by a set of terms $I(T)$ satisfying Girard's conditions.

Definition

- A signature is valid if for all $f : T$, we have $f \in I(T)$.
- $t : T$ is computable if $t \in I(T)$.

Steps

- 1 \mathcal{R} respects SCT \Rightarrow no infinite chains,
- 2 No infinite chains \Rightarrow signature is valid,
- 3 Signature is valid \Rightarrow typable terms are computable.

$\mathcal{R}_f(\mathbb{T}, \mathcal{P}(\mathbb{T}))$ set of partial functions from terms to sets of terms.
Interpretation is defined by the least fixpoint of \mathcal{F} on the strictly inductive poset $\mathcal{R}_f(\mathbb{T}, \mathcal{P}(\mathbb{T}))$.

$$\mathcal{F} : \mathcal{R}_f(\mathbb{T}, \mathcal{P}(\mathbb{T})) \rightarrow \mathcal{R}_f(\mathbb{T}, \mathcal{P}(\mathbb{T}))$$

- \mathcal{F} is increasing,
- For all I , $\text{dom}(\mathcal{F}(I))$ is a reducibility candidate,
- For all I, T , $\mathcal{F}(I)(T)$ is a reducibility candidate.

Requirement

If $T \longrightarrow^* \forall(x : A).B$ then

$$I(T) = \{ t \mid \text{for all } a \in I(A), t a \in I(B [a/x]) \}$$

Example

$$p \ (s \ x) \longrightarrow x$$
$$s \ (p \ x) \longrightarrow x$$
$$f : \mathbb{Z} \Rightarrow \mathbb{Z}$$
$$g : \mathbb{Z} \Rightarrow \mathbb{Z}$$
$$f \ x \longrightarrow g \ x$$
$$g \ 0 \longrightarrow 0$$
$$g \ (s \ x) \longrightarrow f \ x$$
$$g \ (p \ x) \longrightarrow f \ x$$

Implemented in the type-checker *Dedukti*.

Promising results

On 2017 Termination Problem Data Base, in the Higher-Order category, 75 files are proved terminating, including

- recursor of Gödel's system T,
- filter on lists.

<https://github.com/Deducteam/Dedukti/tree/sizechange>

Change order used in SCT [Thiemann, Giesl, 2005] [Coquand, 1992]

`Ord : TYPE`

`0 : Ord`

`S : Ord \Rightarrow Ord`

`lim : ($\mathbb{N} \Rightarrow$ Ord) \Rightarrow Ord`

`ordrec : A \Rightarrow (Ord \Rightarrow A \Rightarrow A) \Rightarrow (($\mathbb{N} \Rightarrow$ Ord) \Rightarrow ($\mathbb{N} \Rightarrow$ A) \Rightarrow A) \Rightarrow Ord \Rightarrow A`

`ordrec x y z (lim f) \rightarrow z f (λ n.ordrec x y z (f n))`

Higher-order matching

$$D (\lambda x. \sin (f x)) \longrightarrow \text{mult} (D (\lambda x. f x)) (\lambda x. \cos (f x))$$

Local growth [Hyvernats, 2013]

$$\begin{aligned} f x & \longrightarrow g (s x) \\ g (s 0) & \longrightarrow 0 \\ g (s (s x)) & \longrightarrow f x \end{aligned}$$