A Perron–Frobenius Theorem for Jordan Blocks for Complexity Proving

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Main result

Theorem (due to René Thiemann):

If $A \in \mathbb{R}_{\geq 0}^{n \times n}$ is a non-negative real square matrix, then one of the largest Jordan blocks of A has the spectral radius ρ_A as the eigenvalue.

Proof

Very nontrivial, but believe it, it's formalized in Isabelle/HOL.

Complexity analysis

- What's the complexity of function sort?

```
sort (x :: xs)     = insert x (sort xs)
sort Nil           = Nil
insert x (y :: ys) = x :: y :: ys        when x <= y
insert x (y :: ys) = y :: insert x ys    when x > y
insert x Nil       = x :: Nil
```

Polynomial interpretation method

Find monotone polynomials $[\![sort]\!] : \mathbb{N} \to \mathbb{N}$, $[\![::]\!] : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$, ..., s.t.

$$[\![sort\ (x :: xs) > insert\ x\ (sort\ xs)]\!]$$
$$[\![sort\ nil > nil]\!]$$
$$[\![insert\ x\ (y :: ys) > x :: y :: ys]\!]$$
$$[\![insert\ x\ (y :: ys) > y :: insert\ x\ ys]\!]$$
$$[\![insert\ x\ nil > x :: nil]\!]$$

Theorem:
If such interpretations exist, then the program terminates, and the runtime is $O\left([\![sort\ \left(x_1 :: (\ldots :: (x_k :: nil)\ )\right)]\!]\right)$

Polynomial runtime demands
$x\ [\![::]\!]\ y = f\ x + y$

Matrix interpretation method

Find affine maps $[\![sort]\!] : \mathbb{N}^n \to \mathbb{N}^n$, $[\![::]\!] : \mathbb{N}^n \to \mathbb{N}^n \to \mathbb{N}^n$, ..., s.t.

$$
\begin{aligned}
&[\![sort\ (x :: xs) && > insert\ x\ (sort\ xs)]\!] \\
&[\![sort\ nil && > nil]\!] \\
&[\![insert\ x\ (y :: ys) && > x :: y :: ys]\!] \\
&[\![insert\ x\ (y :: ys) && > y :: insert\ x\ ys]\!] \\
&[\![insert\ x\ nil && > x :: nil]\!]
\end{aligned}
$$

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} > \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Leftrightarrow \begin{matrix} x_1 > y_1 \\ x_2 \geq y_2 \\ x_3 \geq y_3 \end{matrix}
$$

Theorem:

If such interpretations exist, then the program terminates, and the runtime is $O\left(\left\| [\![sort\ (x_1 :: (\ldots :: (x_k :: nil)\ ))]\!] \right\|\right)$

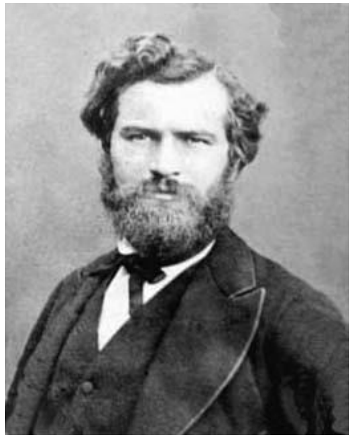
Runtime via matrix power

- Let $[\![sort]\!]\ x = Sx + s$, $x\ [\![::]\!]\ y = Cx + \boldsymbol{A}y + c$, $[\![nil]\!] = n$.

  Then $[\![sort\ (x_1 :: \cdots :: x_k :: nil)]\!]$
  $$= S \cdot \left(Cx_1 + \boldsymbol{A}Cx_2 + \cdots \boldsymbol{A}^{k-1}Cx_k + \boldsymbol{A}^k n\right) + \alpha$$
  $$\in O\left(k \cdot \left\|\boldsymbol{A}^k\right\|\right)$$
- So, if $\left\|\boldsymbol{A}^k\right\| \in O\left(k^d\right)$, runtime is $O\left(k^{d+1}\right)$

Jordan Normal Form precisely gives $\boldsymbol{A}^k$

Jordan normal form (JNF)



- Example:
  - $J = \begin{bmatrix} 2 & 1 & & & & \\ & 2 & 1 & & & \\ & & 2 & & & \\ & & & 3 & 1 & \\ & & & & 3 & \\ & & & & & 4 \end{bmatrix}$

Jordan blocks, generally $\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$

Theorem:

Every square matrix has a JNF (over $\mathbb{C}$), i.e. $A = P\,J\,P^{-1}$

Note: $A^k = P\,J\,P^{-1}\,P\,J\,P^{-1} \cdots P\,J\,P^{-1} = P\,J^k\,P^{-1}$

Power of JNF

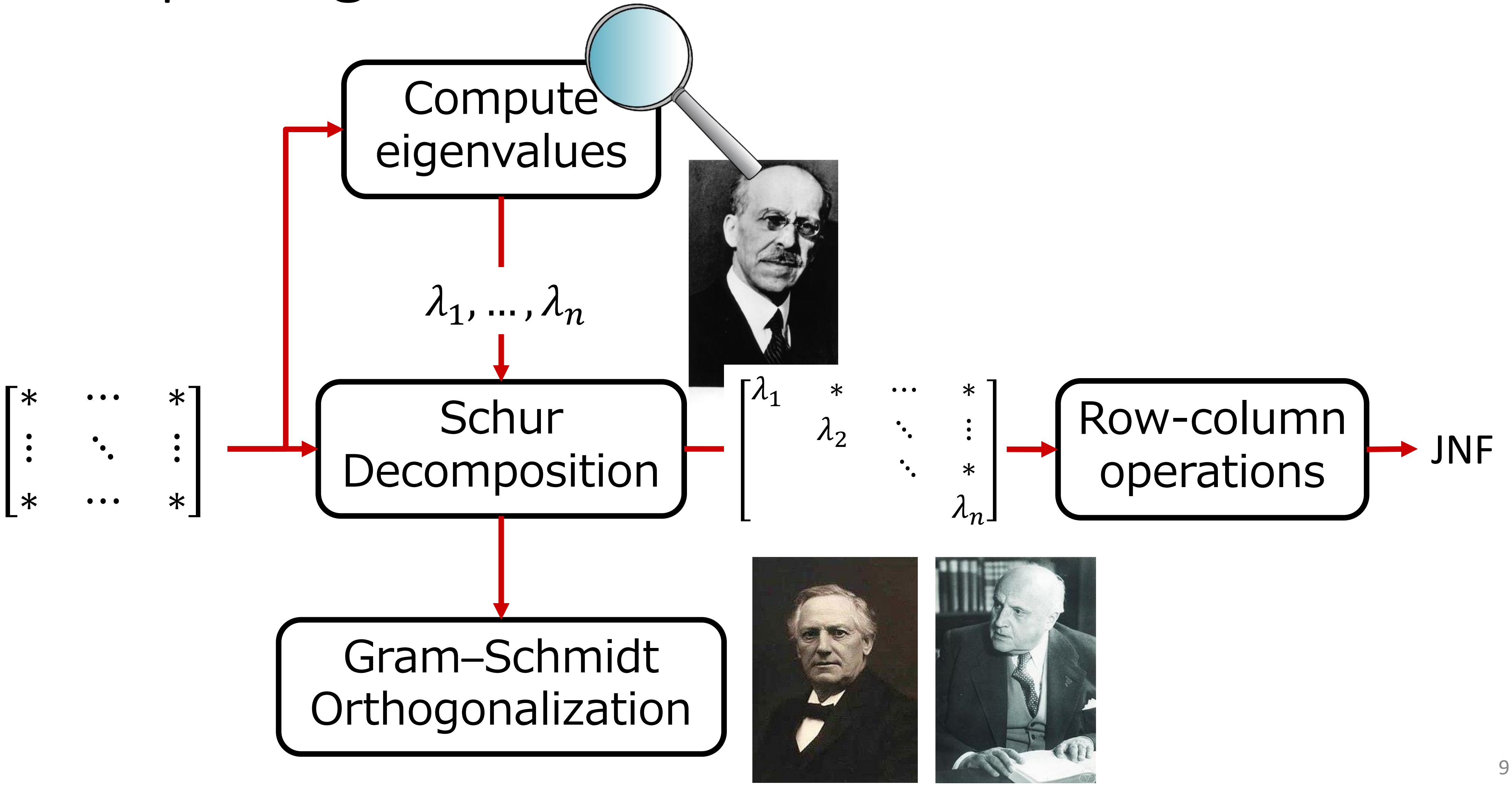
- Example:

$$J^k = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}}^{k} & & \\ & \boxed{\begin{matrix} 3 & 1 \\ & 3 \end{matrix}}^{k} & \\ & & \boxed{4}^{k} \end{bmatrix}^k = \begin{bmatrix} 2^k & 2^{k-1}k & 2^{k-3}k(k-1) & & & \\ & 2^k & 2^{k-1}k & & & \\ & & 2^k & & & \\ & & & 3^k & 3^{k-1}k & \\ & & & & 3^k & \\ & & & & & 4^k \end{bmatrix}$$

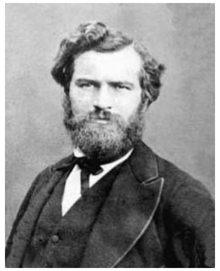
Lemma:

$$\overbrace{\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}}^{\text{size } n}{}^{k} = \begin{bmatrix} \binom{k}{0}\lambda^k & \binom{k}{1}\lambda^{k-1} & \cdots & \binom{k}{n-1}\lambda^{k-n-1} \\ & \binom{k}{0}\lambda^k & \cdots & \binom{k}{n-2}\lambda^{k-n-2} \\ & & \ddots & \vdots \\ & & & \binom{k}{0}\lambda^k \end{bmatrix}$$

Computing JNFs (following [Piziak & Odell '07])



Computing eigenvalues



$A \in \mathbb{Z}^{n \times n}$ → **Gauss–Jordan elimination**

↓ characteristic polynomial $\chi_A \in \mathbb{Z}[x]$

Polynomial factorization

$$\chi_A = f_1 \cdot \cdots \cdot f_m \in \mathbb{Z}[x]$$

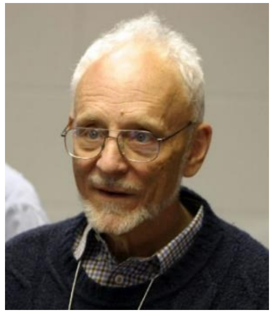
$\deg f_i \leq 2$ → **High-school math** → $\lambda_1, \ldots, \lambda_j \in \mathbb{C}$

→ **Algebraic number representation** → $\lambda_{j+1}, \ldots, \lambda_n \in \mathbb{C}$

Polynomial factorization

$f \in \mathbb{Z}[x]$

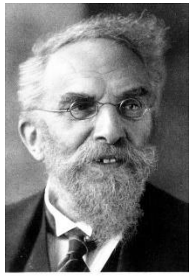
find prime → p → Berlekamp factorization



s.t. coprime p (lead_coeff f), (f mod p) is square free

$$f \equiv g_1 \cdot \cdots \cdot g_l \pmod{p}$$

factor bound → k → Hensel lifting



All Isabelle-formalized, but highly involved and expensive to run!

LLL reconstruction

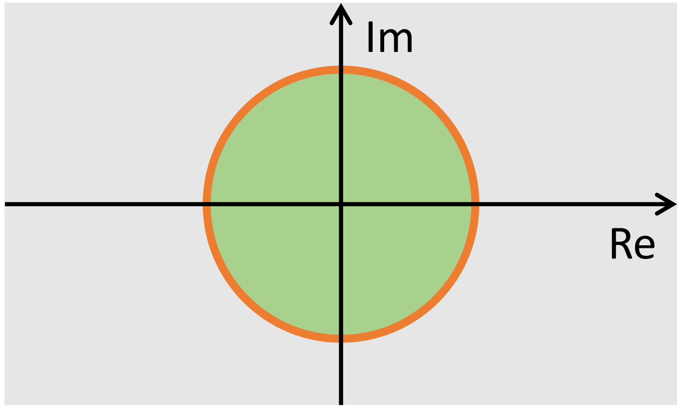
$$f = f_1 \cdot \cdots \cdot f_m \in \mathbb{Z}[x]$$

Power of Jordan blocks, revisited

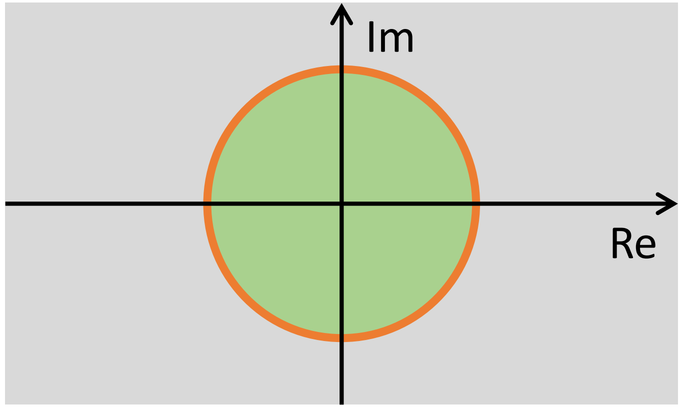
Lemma:

$$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}^k = \begin{bmatrix} \binom{k}{0}\lambda^k & \binom{k}{1}\lambda^{k-1} & \cdots & \binom{k}{n-1}\lambda^{k-n-1} \\ & \binom{k}{0}\lambda^k & \cdots & \binom{k}{n-2}\lambda^{k-n-2} \\ & & \ddots & \vdots \\ & & & \binom{k}{0}\lambda^k \end{bmatrix}$$

- Observation
  - $|\lambda| > 1$ … **EXPTIME**
  - $|\lambda| = 1$ … **need care**
  - $|\lambda| < 1$ … **don't-care**



Checking polynomial growth



input: $A \in \mathbb{N}^{n\times n}, d \in \mathbb{N}$

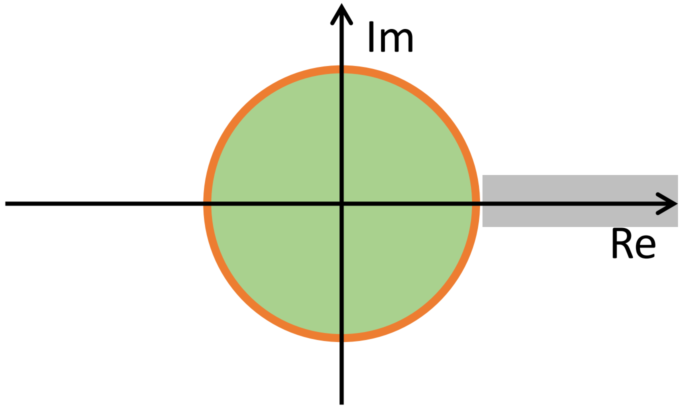
output: accept if $|A^k| \in \mathcal{O}(k^d)$

1. compute all eigenvalues Λ
2. reject (EXPTIME) if $\boldsymbol{\rho_A = \max_{\lambda\in\Lambda}|\lambda| > 1}$
3. compute JNF for each $\boldsymbol{\lambda \in \Lambda}$ **s.t.** $\boldsymbol{|\lambda| = 1}$

Theorem (Perron–Frobenius, basic):
If $A \in \mathbb{R}_{\geq 0}^{n\times n}$ then ρ_A is an eigenvalue of A.
(i.e., $\chi_A(\rho_A) = 0$)

Checking polynomial growth

input: $A \in \mathbb{N}^{n \times n}, d \in \mathbb{N}$
output: accept if $|A^k| \in \mathcal{O}(k^d)$



1. reject (EXPTIME) if χ_A has root in $(1, \infty)$
2. compute **all eigenvalues** Λ
3. compute JNF for each $\boldsymbol{\lambda \in \Lambda}$ **s.t.** $|\boldsymbol{\lambda}| = \mathbf{1}$

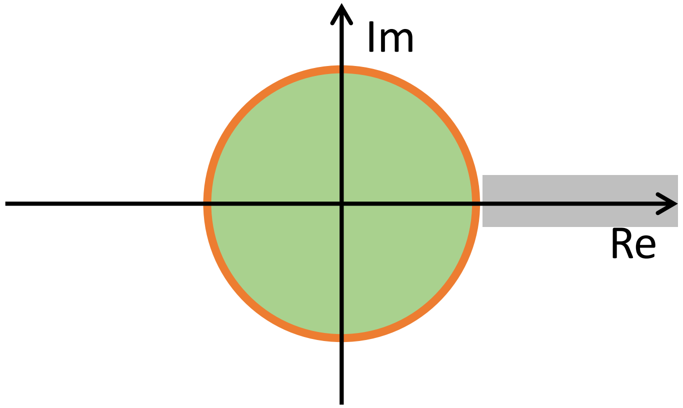
Theorem (Perron–Frobenius, more):
If $A \in \mathbb{R}_{\geq 0}^{n \times n}$ then

$$\chi_A(\lambda) = f(\lambda) \cdot \prod_{i \in I} (\lambda^i - \rho_A^i)$$

with $f(\lambda) = 0 \Longrightarrow |\lambda| < \rho_A$

Checking polynomial growth

input: $A \in \mathbb{N}^{n \times n}$, $d \in \mathbb{N}$
output: accept if $|A^k| \in \mathcal{O}(k^d)$



1. reject (EXPTIME) if χ_A has root in $(\mathbf{1}, \infty)$
2. compute **all eigenvalues** Λ
3. compute JNF for each **$\lambda \in \Lambda$ s.t. $|\lambda| = 1$**

Theorem (Perron–Frobenius, more):

If $A \in \mathbb{R}_{\geq 0}^{n \times n}$ and $\rho_A \leq 1$ then

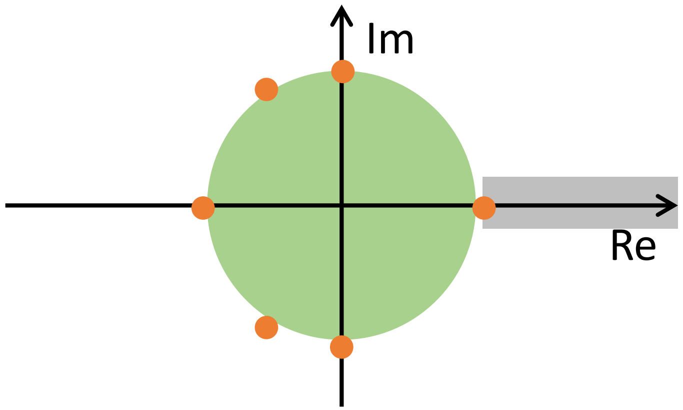
$$\chi_A(\lambda) = f(\lambda) \cdot \prod_{i \in I} (\lambda^i - 1)$$

with $f(\lambda) = 0 \Longrightarrow |\lambda| < 1$

concerned eigenvals are roots of unity!

Checking polynomial growth

input: $A \in \mathbb{N}^{n \times n}$, $d \in \mathbb{N}$
output: accept if $|A^k| \in \mathcal{O}(k^d)$



1. reject (EXPTIME) if χ_A has root in $(\mathbf{1}, \infty)$
2. compute **all eigenvalues** Λ
3. compute JNF for ***i*-th root of unity** up to some i

Theorem (Perron–Frobenius, more):

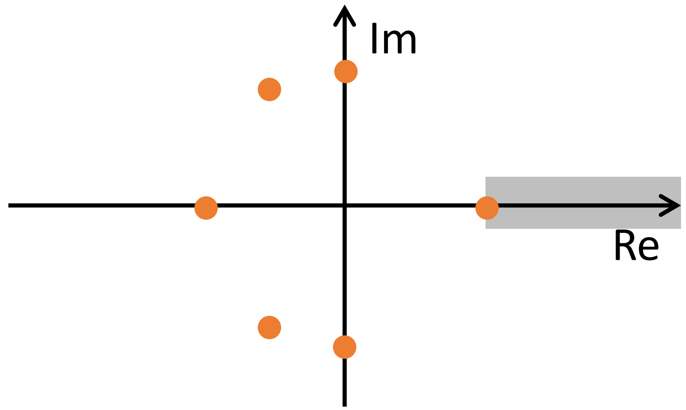
If $A \in \mathbb{R}_{\geq 0}^{n \times n}$ and $\rho_A \leq 1$ then

$$\chi_A(\lambda) = f(\lambda) \cdot \prod_{i \in I} (\lambda^i - 1)$$

with $f(\lambda) = 0 \Longrightarrow |\lambda| < 1$ **can be ignored!**

Checking polynomial growth

input: $A \in \mathbb{N}^{n \times n}$, $d \in \mathbb{N}$
output: accept if $|A^k| \in \mathcal{O}(k^d)$



1. reject (EXPTIME) if χ_A has root in $(1, \infty)$

2. compute JNF for ***i*-th root of unity** up to some i
   - 1,
   - -1,
   - $\frac{-1+\sqrt{3}\mathbb{i}}{2}, \frac{-1-\sqrt{3}\mathbb{i}}{2}$
   - …

not very nice!

Power of JNF, revisited

- Example:

$$J^k = \begin{bmatrix} \begin{bmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{bmatrix}^k & & \\ & \begin{bmatrix} \lambda_2 & 1 \\ & \lambda_2 \end{bmatrix}^k & \\ & & [\lambda_3]^k \end{bmatrix}^k$$

Only the largest Jordan block matter...

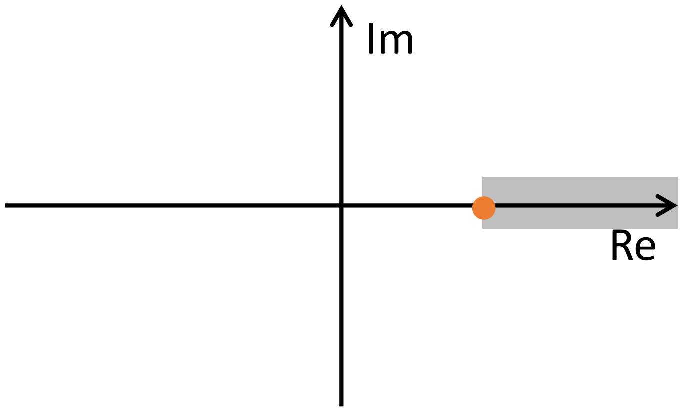
Lemma: If $|\lambda| = 1$ then

$$\underbrace{\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}}_{\text{size } n}^k \in \mathcal{O}\left((n-1)^k\right)$$

Theorem (Perron–Frobenius–Thiemann):
If $A \in \mathbb{R}_{\geq 0}^{n \times n}$, then one of the largest Jordan Blocks of A has ρ_A as the eigenvalue.

Finally!

input: $A \in \mathbb{N}^{n \times n}$, $d \in \mathbb{N}$
output: accept if $|A^k| \in \mathcal{O}(k^d)$



1. reject (EXPTIME) if χ_A has root in $(\mathbf{1}, \infty)$

2. compute JNF for eigenvalue **1**

Corollary:
For $A \in \mathbb{R}_{\geq 0}^{n \times n}$, $\|A^k\| \in \mathcal{O}(k^d)$ iff
- no eigenvalue in $(1, \infty)$ and
- the Jordan block of A for eigenvalue 1 is of size at most $d + 1$.

Conclusion

- Contribution
  - a very nontrivial Perron–Frobenius style theorem
  - all formalized in IsaFoR, implemented in CeTA
  - On 6,690 matrices from past TermComps, x5 speed up in certification.
- Future work
  - Utilize in complexity analysis (not only certification)?