Inference of Linear Upper-Bounds on the Expected Cost by Solving Cost Relations

Alicia Merayo and Samir Genaim

Complutense University of Madrid

Oxford, 07/19/2018

Overview

Introduction

Probabilistic programs

Cost Analysis of Probabilistic Programs

Conclusions

Cost Analysis

Cost Analysis

Application of static techniques in order to determine the amount of resources that are needed to execute a program.

- Runtime (execution steps)
- Memory consumption
- Generic cost model: ticks

Cost Analysis

Cost Analysis

Application of static techniques in order to determine the amount of resources that are needed to execute a program.

- Runtime (execution steps)
- Memory consumption
- Generic cost model: ticks

Cost Analysis

- **Runtime**: Sort

```
for (i = 0; i < length(a); i++){
  val = a[i];
  j = i-1;
  while(j > 0 && a[j] > val){
    a[j+1] = a[j];
    j = j-1;
    tick(1);
  }
  a[j+1] = val;
  tick(1);
}
```

- **Worst case cost**:

$$\frac{\text{length}(a)^2 + \text{length}(a)}{2}$$

- **Memory**: Dynamic Array

```
while(! empty(list)){
  if list[i] > 0{
    if (j>=n){
      enlarge(a,l,n);
      tick(2*l*n);
    }
    a[j] = head(list);
    list = tail(list);
    j = j+1;
  }
  i = i+1;
}
```

- **Worst case cost**:

$$2 * l * n * (\mathit{length}(\mathit{list}))$$

Cost Analysis

- **Runtime**: Sort

```
for (i = 0; i < length(a); i++){
  val = a[i];
  j = i-1;
  while(j > 0 && a[j] > val){
    a[j+1] = a[j];
    j = j-1;
    tick(1);
  }
  a[j+1] = val;
  tick(1);
}
```

- **Worst case cost**:

$$\frac{\text{length}(a)^2 + \text{length}(a)}{2}$$

- **Memory**: Dynamic Array

```
while(! empty(list)){
  if list[i] > 0{
    if (j>=n){
      enlarge(a,l,n);
      tick(2*l*n);
    }
    a[j] = head(list);
    list = tail(list);
    j = j+1;
  }
  i = i+1;
}
```

- **Worst case cost**:

$$2 * l * n * (\mathit{length}(\mathit{list}))$$

Cost Analysis

- **Runtime**: Sort

```
for (i = 0; i < length(a); i++){
  val = a[i];
  j = i-1;
  while(j > 0 && a[j] > val){
    a[j+1] = a[j];
    j = j-1;
    tick(1);
  }
  a[j+1] = val;
  tick(1);
}
```

- **Worst case cost**:

$$\frac{\text{length}(a)^2 + \text{length}(a)}{2}$$

- **Memory**: Dynamic Array

```
while(! empty(list)){
  if list[i] > 0{
    if (j>=n){
      enlarge(a,l,n);
      tick(2*l*n);
    }
    a[j] = head(list);
    list = tail(list);
    j = j+1;
  }
  i = i+1;
}
```

- **Worst case cost**:

$$2 * l * n * (\mathit{length}(\mathit{list}))$$

Cost Analysis

- **Runtime**: Sort

```
for (i = 0; i < length(a); i++){
  val = a[i];
  j = i-1;
  while(j > 0 && a[j] > val){
    a[j+1] = a[j];
    j = j-1;
    tick(1);
  }
  a[j+1] = val;
  tick(1);
}
```

- **Worst case cost**:

$$\frac{\text{length}(a)^2 + \text{length}(a)}{2}$$

- **Memory**: Dynamic Array

```
while(! empty(list)){
  if list[i] > 0{
    if (j>=n){
      enlarge(a,l,n);
      tick(2*l*n);
    }
    a[j] = head(list);
    list = tail(list);
    j = j+1;
  }
  i = i+1;
}
```

- **Worst case cost**:

$$2 * l * n * (\mathit{length}(\mathit{list}))$$

SACO

SACO

Static analyzer for automatically inferring upper/lower bounds on the worst/best-case resource consumption of ABS programs.
`http://costa.fdi.ucm.es/saco/web/`

ABS

- Functional + imperative.
- Based on concurrent objects.

SACO

SACO

Static analyzer for automatically inferring upper/lower bounds on the worst/best-case resource consumption of ABS programs.
`http://costa.fdi.ucm.es/saco/web/`

ABS

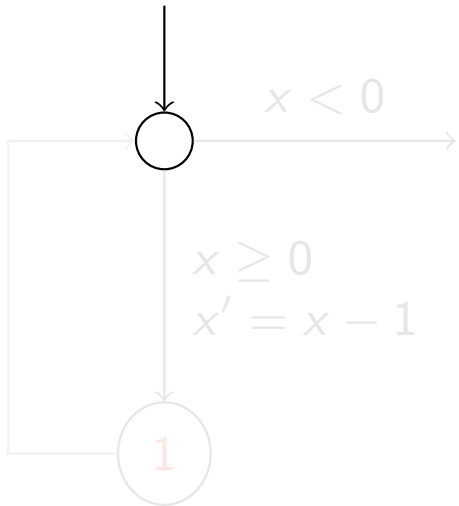
- Functional + imperative.
- Based on concurrent objects.

Cost Analysis using SACO

```
while(x>=0){
  x = x-1;
  tick(1);
}
```

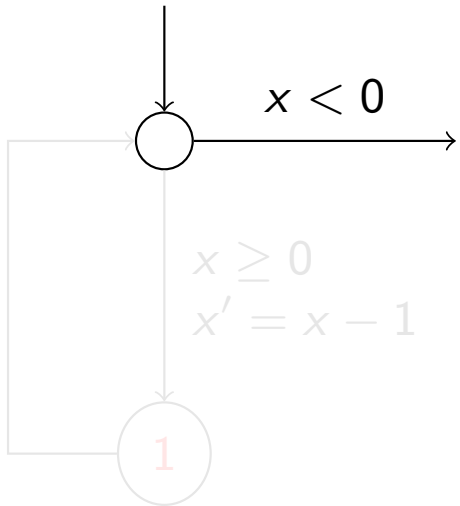
Cost Analysis using SACO

```
while(x>=0){
  x = x-1;
  tick(1);
}
```



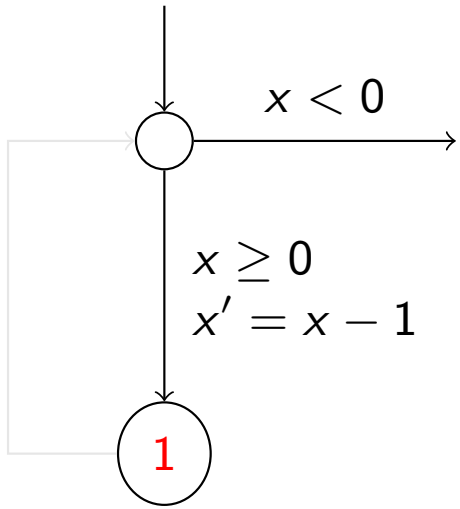
Cost Analysis using SACO

```
while(x>=0){
  x = x-1;
  tick(1);
}
```



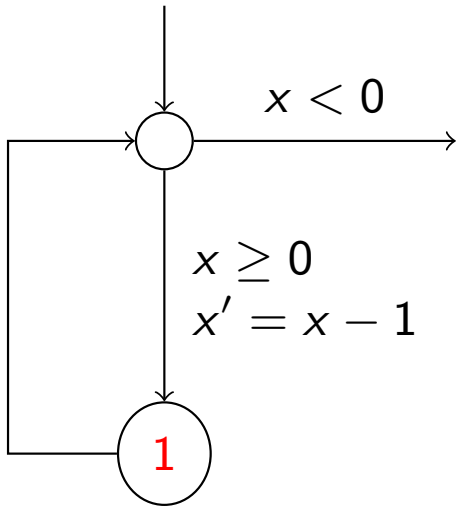
Cost Analysis using SACO

```
while(x>=0){
  x = x-1;
  tick(1);
}
```

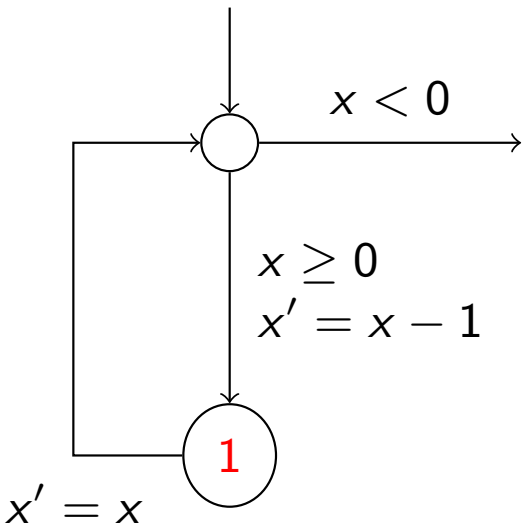


Cost Analysis using SACO

```
while(x>=0){
  x = x-1;
  tick(1);
}
```



Cost Analysis using SACO



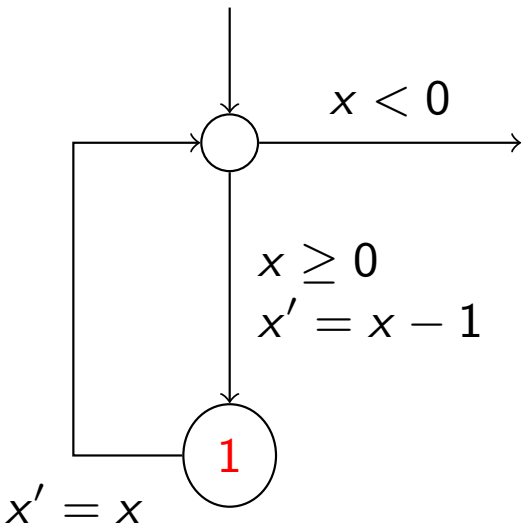
- From CFG to Cost Relations

$$
\begin{array}{ll}
C(x) = 0 & \{x < 0\} \\
C(x) = D(x') & \{x >= 0, x' = x - 1\} \\
D(x) = 1 + C(x') & \{x' = x\}
\end{array}
$$

- Simplifying cost relations

$$
\begin{array}{ll}
C(x) = 0 & \{x < 0\} \\
C(x) = 1 + C(x') & \{x \geq 0,\ x' = x - 1\}
\end{array}
$$

Cost Analysis using SACO



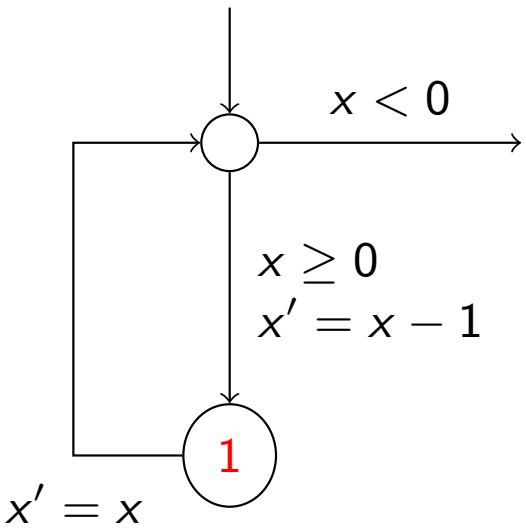
- **From CFG to Cost Relations**

$$\begin{array}{ll} C(x) = 0 & \{x < 0\} \\ C(x) = D(x') & \{x >= 0, x' = x - 1\} \\ D(x) = 1 + C(x') & \{x' = x\} \end{array}$$

- **Simplifying cost relations**

$$\begin{array}{ll} C(x) = 0 & \{x < 0\} \\ C(x) = 1 + C(x') & \{x \geq 0,\ x' = x - 1\} \end{array}$$

Cost Analysis using SACO



- **From CFG to Cost Relations**

$$
\begin{array}{ll}
C(x) = 0 & \{x < 0\} \\
C(x) = D(x') & \{x >= 0, x' = x - 1\} \\
D(x) = 1 + C(x') & \{x' = x\}
\end{array}
$$

- **Simplifying cost relations**

$$
\begin{array}{ll}
C(x) = 0 & \{x < 0\} \\
C(x) = 1 + C(x') & \{x \geq 0,\ x' = x - 1\}
\end{array}
$$

Cost Analysis using SACO

Cost relations are like recurrence relations, but the have some difference like nondeterminism.

```
while(x>=0){
  x = x-1;
  tick(1);
}
```

$$\begin{array}{ll} C(x) = 0 & \{x < 0\} \\ C(x) = 1 + C(x') & \{x \geq 0,\ x' = x - 1\} \end{array}$$

Cost Analysis using SACO

Cost relations are like recurrence relations, but the have some difference like nondeterminism.

```
while(x>=0){
  x = x-1;
  tick(1);
}
```

$$\begin{array}{ll} C(x) = 0 & \{x < 0\} \\ C(x) = 1 + C(x') & \{x \geq 0,\ x' = x - 1\} \end{array}$$

```
while(x>=0){
  if(*)
    x = x-1;
  else
    x = x-2;
  tick(1);
}
```

$$\begin{array}{ll} C(x) = 0 & \{x < 0\} \\ C(x) = 1 + C(x') & \{x \geq 0,\ x' = x - 1\} \\ C(x) = 1 + C(x') & \{x \geq 0,\ x' = x - 2\} \end{array}$$

Cost Analysis using SACO

Upper Bound of a Cost Relation

Upper-bound on the set of values to which a query might evaluate. It is an upper bound of the worst case of the execution.

```
while(x>=0){
  if(*)
    x = x-1;
  else
    x = x-2;
  tick(1);
}
```

$$
\begin{array}{ll}
C(x) = 0 & \{x < 0\} \\
C(x) = 1 + C(x') & \{x \geq 0,\ x' = x - 1\} \\
C(x) = 1 + C(x') & \{x \geq 0,\ x' = x - 2\}
\end{array}
$$

$$C(10) = 10, 9, 8, \ldots$$

Cost Analysis using SACO

Upper Bound of a Cost Relation

Upper-bound on the set of values to which a query might evaluate. It is an upper bound of the worst case of the execution.

```
while(x>=0){
  if(*)
    x = x-1;
  else
    x = x-2;
  tick(1);
}
```

$$
\begin{array}{ll}
C(x) = 0 & \{x < 0\} \\
C(x) = 1 + C(x') & \{x \geq 0,\ x' = x - 1\} \\
C(x) = 1 + C(x') & \{x \geq 0,\ x' = x - 2\}
\end{array}
$$

$$C(10) = 10, 9, 8, \ldots$$

Cost Analysis using SACO

Upper Bound of a Cost Relation

Upper-bound on the set of values to which a query might evaluate. It is an upper bound of the worst case of the execution.

```
while(x>=0){
  if(*)
    x = x-1;
  else
    x = x-2;
  tick(1);
}
```

$$
\begin{array}{ll}
C(x) = 0 & \{x < 0\} \\
C(x) = 1 + C(x') & \{x \geq 0,\ x' = x - 1\} \\
C(x) = 1 + C(x') & \{x \geq 0,\ x' = x - 2\}
\end{array}
$$

$$C(10) = 10, 9, 8, \ldots$$

When worst-case is not enough...

```
while(msgs != null){
  if(send(head(data))){
    msgs = tail(msgs);
  }
  tick(1);
}
```

$$
\begin{array}{ll}
C(n) = 0 & \{n \leq 0\} \\
C(n) = 1 + C(n') & \{n \geq 1, n' = n - 1\} \\
C(n) = 1 + C(n') & \{n \geq 1, n' = n\}
\end{array}
$$

- Infinite complexity
- What happens if we trust the quality of the network to some degree?

When worst-case is not enough...

```
while(msgs != null){
  if(send(head(data))){
    msgs = tail(msgs);
  }
  tick(1);
}
```

$$
\begin{array}{ll}
C(n) = 0 & \{n \leq 0\} \\
C(n) = 1 + C(n') & \{n \geq 1, n' = n - 1\} \\
C(n) = 1 + C(n') & \{n \geq 1, n' = n\}
\end{array}
$$

- Infinite complexity
- What happens if we trust the quality of the network to some degree?

When worst-case is not enough...

```
while(msgs != null){
  if(send(head(data))){
    msgs = tail(msgs);
  }
  tick(1);
}
```

$$
\begin{array}{ll}
C(n) = 0 & \{n \leq 0\} \\
C(n) = 1 + C(n') & \{n \geq 1, n' = n - 1\} \\
C(n) = 1 + C(n') & \{n \geq 1, n' = n\}
\end{array}
$$

- Infinite complexity
- What happens if we trust the quality of the network to some degree?

When worst-case is not enough...

```
while(msgs != null){
  if(send(head(data))){
    msgs = tail(msgs);
  }
  tick(1);
}
```

$$
\begin{array}{ll}
C(n) = 0 & \{n \leq 0\} \\
C(n) = 1 + C(n') & \{n \geq 1, n' = n - 1\} \\
C(n) = 1 + C(n') & \{n \geq 1, n' = n\}
\end{array}
$$

- Infinite complexity
- What happens if we trust the quality of the network to some degree?

Probabilistic programs

Addition of probabilistic choices to the set of instructions:

$$\text{inst}_1 \oplus_p \text{inst}_2$$

where

- $p \in [0, 1]$
- inst_1 and inst_2 will execute with probability p and $1 - p$, respectively.
- It can be generalized such that inst_i is executed with probability p_i, and $\sum_i p_i = 1$

$$\begin{array}{l} \text{while}(x < y)\{ \\ \quad x = x - 1 \oplus_{\frac{1}{3}} y = y - 2; \\ \quad \text{tick}(1); \\ \} \end{array}$$

Probabilistic programs

Addition of probabilistic choices to the set of instructions:

$$\text{inst}_1 \oplus_p \text{inst}_2$$

where

- $p \in [0, 1]$
- inst_1 and inst_2 will execute with probability p and $1 - p$, respectively.
- It can be generalized such that inst_i is executed with probability p_i, and $\sum_i p_i = 1$

$$\begin{array}{l} \text{while}(x < y)\{ \\ \quad x = x - 1 \oplus_{\frac{1}{3}} y = y - 2; \\ \quad \text{tick}(1); \\ \} \end{array}$$

Probabilistic programs

Addition of probabilistic choices to the set of instructions:

$$\text{inst}_1 \oplus_p \text{inst}_2$$

where

- $p \in [0, 1]$
- inst_1 and inst_2 will execute with probability p and $1 - p$, respectively.
- It can be generalized such that inst_i is executed with probability p_i, and $\sum_i p_i = 1$

$$\begin{array}{l} \text{while}(x < y)\{ \\ \quad x = x - 1 \oplus_{\frac{1}{3}} y = y - 2; \\ \quad \text{tick}(1); \\ \} \end{array}$$

Probabilistic programs

Addition of probabilistic choices to the set of instructions:

$$\text{inst}_1 \oplus_p \text{inst}_2$$

where

- $p \in [0, 1]$
- inst_1 and inst_2 will execute with probability p and $1 - p$, respectively.
- It can be generalized such that inst_i is executed with probability p_i, and $\sum_i p_i = 1$

$$\begin{array}{l} \text{while}(x < y)\{ \\ \quad x = x - 1 \oplus_{\frac{1}{3}} y = y - 2; \\ \quad \text{tick}(1); \\ \} \end{array}$$

Expected cost

Packages transmission on a net

We have got a net for transmitting packages on which the probability of succeeding is $\frac{2}{3}$, and we want to transmit n packages.

```
while(msgs != null){
  if(send(head(data)))
    msgs = tail(msgs);
  tick(1);
}
```

$\Longrightarrow$

```
while (n > 0){
  n = n - 1; ⊕_{2/3} skip;
  tick(1);
}
```

Expected cost

Packages transmission on a net

We have got a net for transmitting packages on which the probability of succeeding is $\frac{2}{3}$, and we want to transmit n packages.

```
while(msgs != null){
  if(send(head(data)))
    msgs = tail(msgs);
  tick(1);
}
```

$\Longrightarrow$

```
while (n > 0){
  n = n − 1; ⊕_{2/3} skip;
  tick(1);
}
```

Expected cost

Packages transmission on a net

We have got a net for transmitting packages on which the probability of succeeding is $\frac{2}{3}$, and we want to transmit n packages.

```
while(msgs != null){
  if(send(head(data)))
    msgs = tail(msgs);
  tick(1);
}
```

$\Longrightarrow$

```
while (n > 0){
  n = n - 1; ⊕_{2/3} skip;
  tick(1);
}
```

Expected cost

Expected cost

Let

- S be an infinite sequence of independent choices for the probabilistic operation.
- $\texttt{cost}(n, S)$ be the cost of the execution (with input n) when taking the choices indicated in S.
- $\texttt{Pr}(S)$ be the probability of taking such choices

then the *expected cost* is defined as

$$\sum_{S} \texttt{Pr}(S) \cdot \texttt{cost}(n, S)$$

Expected cost

Expected cost

Let

- S be an infinite sequence of independent choices for the probabilistic operation.
- $\texttt{cost}(n, S)$ be the cost of the execution (with input n) when taking the choices indicated in S.
- $\texttt{Pr}(S)$ be the probability of taking such choices

then the *expected cost* is defined as

$$\sum_{S} \texttt{Pr}(S) \cdot \texttt{cost}(n, S)$$

Expected cost

Expected cost

Let

- S be an infinite sequence of independent choices for the probabilistic operation.
- $\texttt{cost}(n, S)$ be the cost of the execution (with input n) when taking the choices indicated in S.
- $\texttt{Pr}(S)$ be the probability of taking such choices

then the *expected cost* is defined as

$$\sum_{S} \texttt{Pr}(S) \cdot \texttt{cost}(n, S)$$

Expected cost

Expected cost

Let

- S be an infinite sequence of independent choices for the probabilistic operation.
- $\texttt{cost}(n, S)$ be the cost of the execution (with input n) when taking the choices indicated in S.
- $\texttt{Pr}(S)$ be the probability of taking such choices

then the *expected cost* is defined as

$$\sum_S \texttt{Pr}(S) \cdot \texttt{cost}(n, S)$$

Expected cost

Expected cost

Let

- S be an infinite sequence of independent choices for the probabilistic operation.
- $\texttt{cost}(n, S)$ be the cost of the execution (with input n) when taking the choices indicated in S.
- $\texttt{Pr}(S)$ be the probability of taking such choices

then the *expected cost* is defined as

$$\sum_{S} \texttt{Pr}(S) \cdot \texttt{cost}(n, S)$$

Expected cost

$$\sum_{S} \mathtt{Pr}(S) \cdot \mathtt{cost}(n, S)$$

The sum is over all possible sequences S.

- $\sum_S \mathtt{Pr}(S) = 1$

while $(n > 0)\{$

$n = n - 1;\ \oplus_{\frac{2}{3}}$ skip;

tick(1);

$\}$

Expected cost

$$\sum_{S} \mathtt{Pr}(S) \cdot \mathtt{cost}(n, S)$$

- The sum is over all possible sequences S.
- $\sum_S \mathtt{Pr}(S) = 1$

while $(n > 0)\{$

$n = n - 1;\ \oplus_{\frac{2}{3}}$ skip;

tick(1);

$\}$

Expected cost

$$\sum_{S} \mathtt{Pr}(S) \cdot \mathtt{cost}(n, S)$$

- The sum is over all possible sequences S.
- $\sum_S \mathtt{Pr}(S) = 1$

while $(n > 0)\{$

$n = n - 1;\ \oplus_{\frac{2}{3}}$ skip;

tick(1);

$\}$

Expected cost

$$\sum_{S} \mathtt{Pr}(S) \cdot \mathtt{cost}(n, S)$$

- The sum is over all possible sequences S.
- $\sum_S \mathtt{Pr}(S) = 1$

while $(n > 0)\{$

$n = n - 1;\ \oplus_{\frac{2}{3}}$ skip;

tick(1);

$\}$

Expected cost by CRs

```
while (n > 0){
    n = n − 1; ⊕_{2/3} skip;
    tick(1);
}
```

By definition, the expected cost is

$$\sum_{S} \mathtt{Pr}(S) \cdot \mathtt{cost}(n, S)$$

CRs:

$$C(n) = 0 \qquad \{n \leq 0\}$$

$$C(n) = 1 + \frac{2}{3}C(n') + \frac{1}{3}C(n'') \qquad \{n \geq 1, n' = n-1, n'' = n\}$$

Expected cost: $\frac{3}{2}n$

Expected cost by CRs

```
while (n > 0){
    n = n − 1; ⊕_{2/3} skip;
    tick(1);
}
```

By definition, the expected cost is

$$\sum_{S} \mathtt{Pr}(S) \cdot \mathtt{cost}(n, S)$$

CRs:

$$C(n) = 0 \qquad \{n \leq 0\}$$

$$C(n) = 1 + \frac{2}{3}C(n') + \frac{1}{3}C(n'') \qquad \{n \geq 1, n' = n-1, n'' = n\}$$

Expected cost: $\frac{3}{2}n$

Expected cost by CRs

```
while (n > 0){
  n = n − 1; ⊕_{2/3} skip;
  tick(1);
}
```

By definition, the expected cost is

$$\sum_{S} \mathtt{Pr}(S) \cdot \mathtt{cost}(n, S)$$

CRs:

$$C(n) = 0 \qquad \{n \leq 0\}$$

$$C(n) = 1 + \frac{2}{3}C(n') + \frac{1}{3}C(n'') \qquad \{n \geq 1, n' = n - 1, n'' = n\}$$

Expected cost: $\frac{3}{2}n$

Expected cost by CRs

```
while (n > 0){
  n = n − 1; ⊕_{2/3} skip;
  tick(1);
}
```

By definition, the expected cost is

$$\sum_{S} \mathtt{Pr}(S) \cdot \mathtt{cost}(n, S)$$

CRs:

$$C(n) = 0 \qquad \{n \leq 0\}$$

$$C(n) = 1 + \frac{2}{3}C(n') + \frac{1}{3}C(n'') \qquad \{n \geq 1, n' = n - 1, n'' = n\}$$

Expected cost: $\frac{3}{2}n$

Our goal

We want to introduce probabilities into SACO.

Steps

1. Introduction of probabilities into CFGs.
2. Development of techniques for inferring expected cost of CFGs via CRs.
3. Specify probabilities directly in ABS.

Our goal

We want to introduce probabilities into SACO.

Steps

- Introduction of probabilities into CFGs.
- Development of techniques for inferring expected cost of CFGs via CRs.
- Specify probabilities directly in ABS.

Our goal

We want to introduce probabilities into SACO.

Steps

- Introduction of probabilities into CFGs.
- Development of techniques for inferring expected cost of CFGs via CRs.
- Specify probabilities directly in ABS.

Our goal

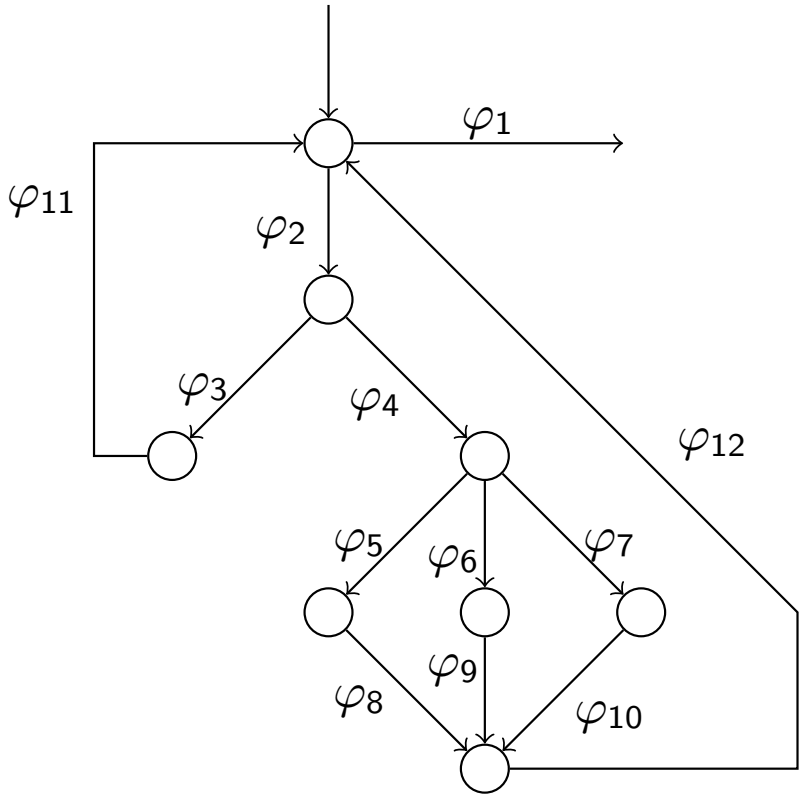
We want to introduce probabilities into SACO.

Steps

- Introduction of probabilities into CFGs.
- Development of techniques for inferring expected cost of CFGs via CRs.
- Specify probabilities directly in ABS.

CFGs with probabilities

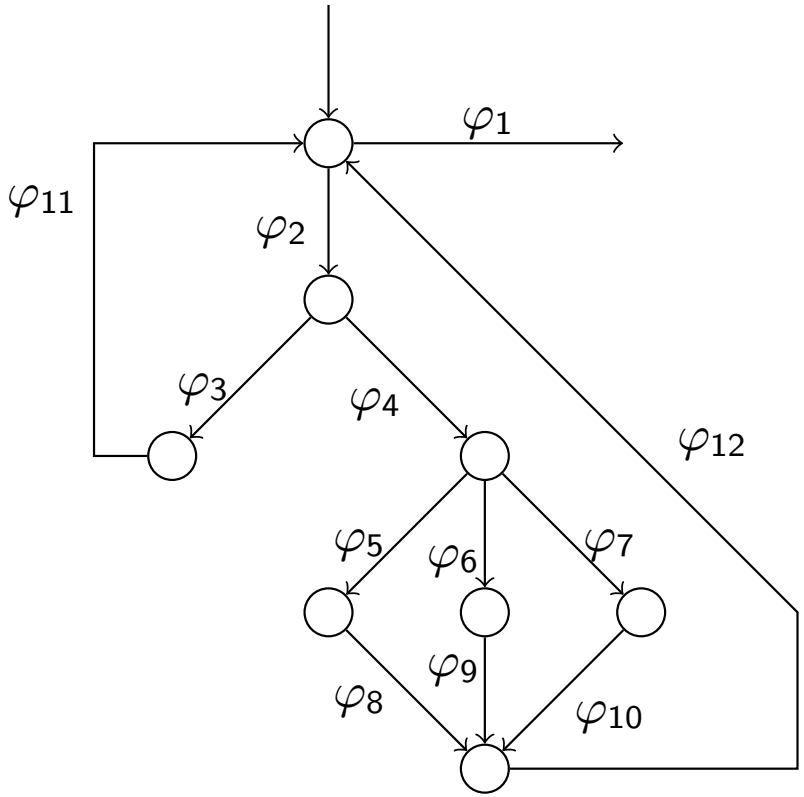
CFG without probabilities:



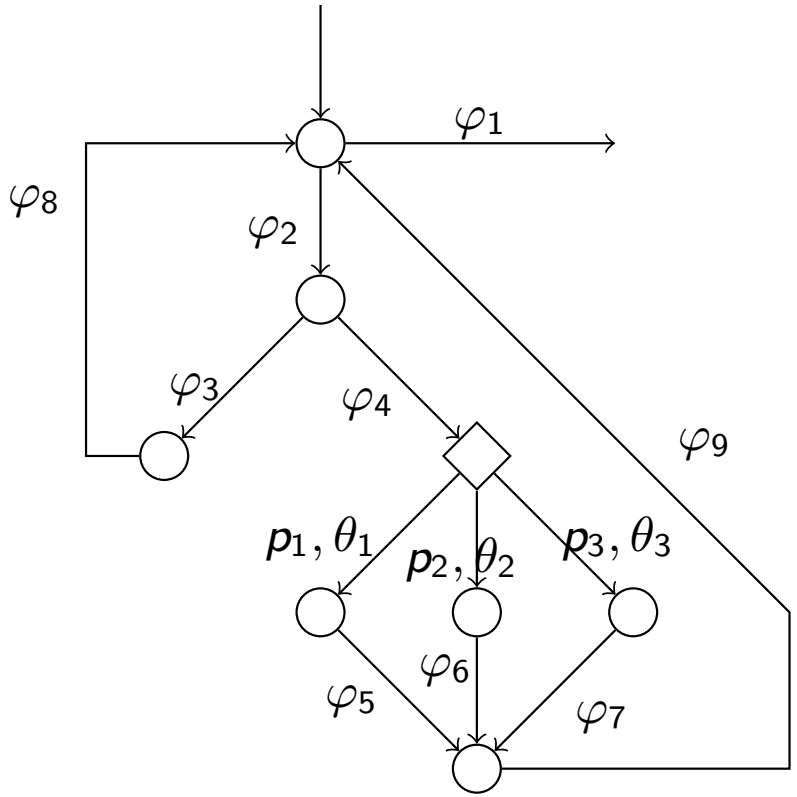
CFG with probabilities:

CFGs with probabilities

CFG without probabilities:

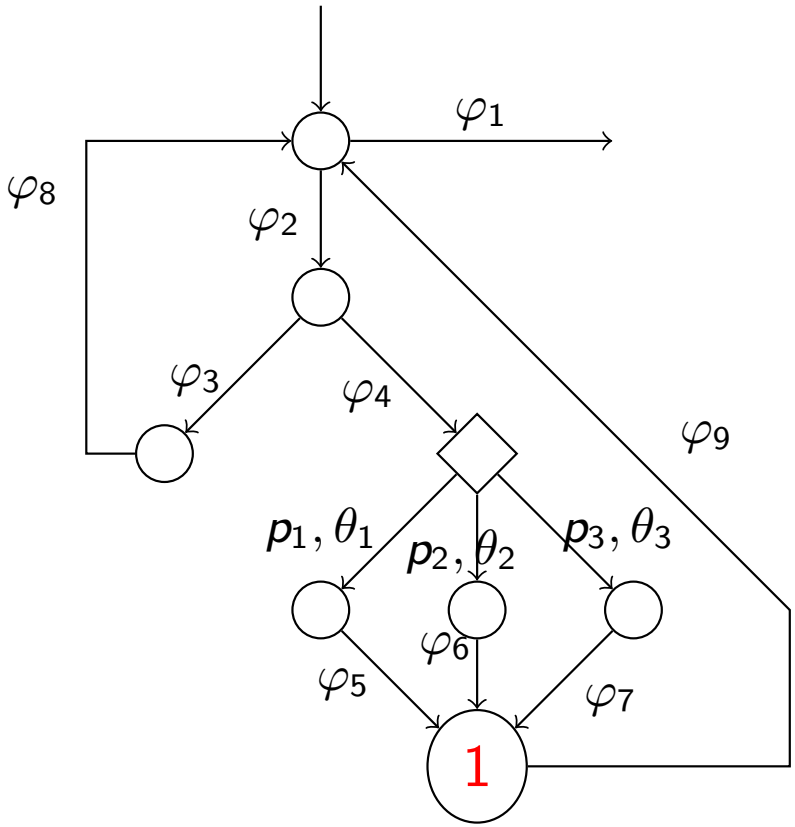


CFG with probabilities:



CFGs with probabilities

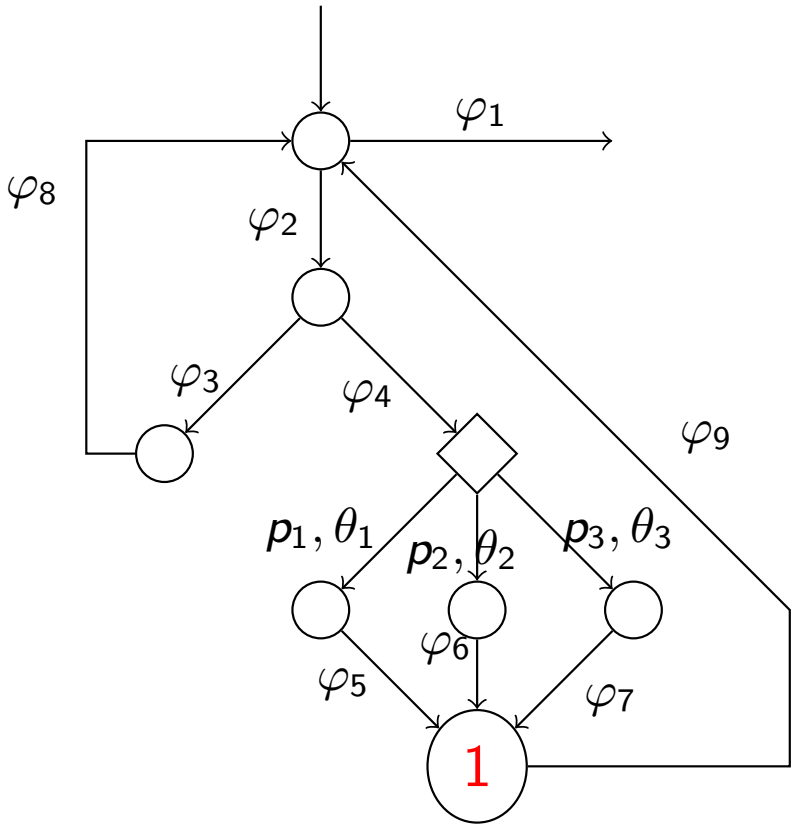
CFG with probabilities:



- Regular instructions represented by circles.
- Probability instructions represented by squares.
- Edges with updates and
  - Probabilities
  - Constraints
- Nodes with cost.
- $\sum_i p_i = 1$
- Choices in normal nodes cover the whole state space.

CFGs with probabilities

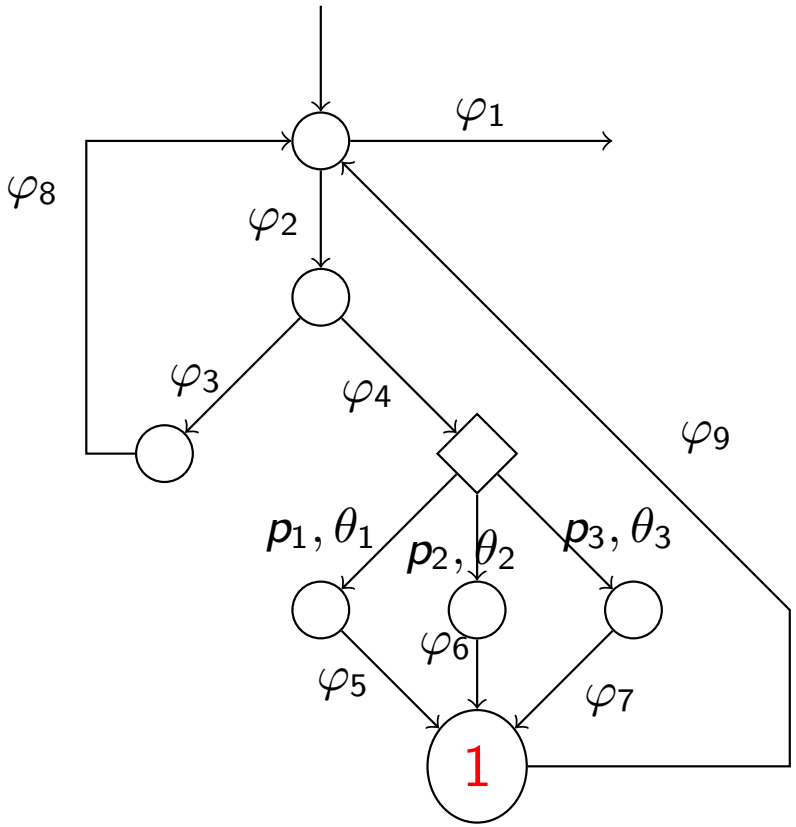
CFG with probabilities:



- Regular instructions represented by circles.
- Probability instructions represented by squares.
- Edges with updates and
  - Probabilities
  - Constraints
- Nodes with cost.
- $\sum_i p_i = 1$
- Choices in normal nodes cover the whole state space.

CFGs with probabilities

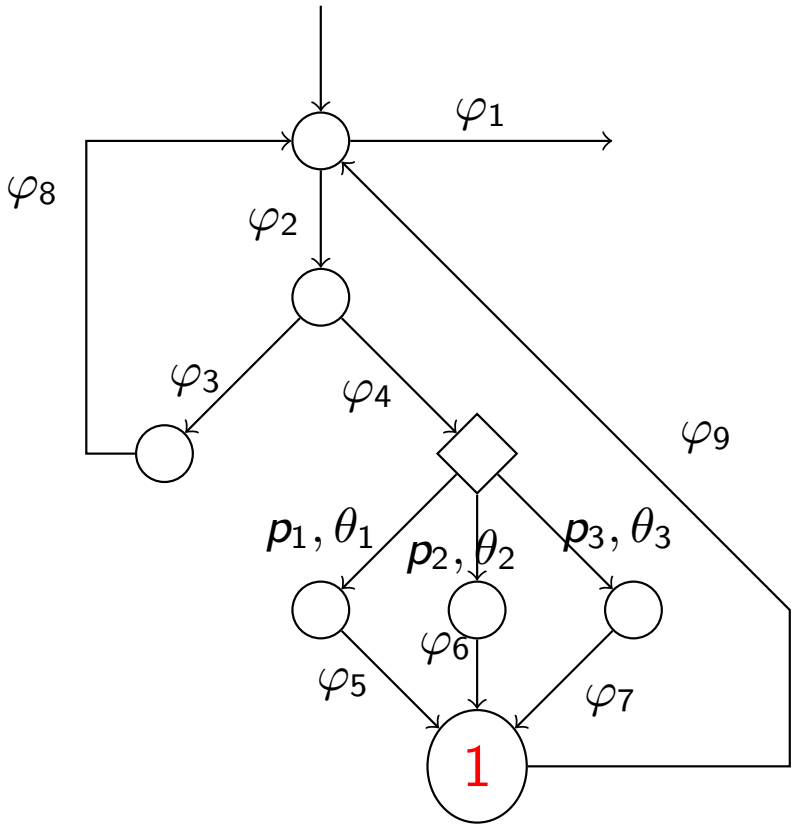
CFG with probabilities:



- Regular instructions represented by circles.
- Probability instructions represented by squares.
- Edges with updates and
  - Probabilities
  - Constraints
- Nodes with cost.
- $\sum_i p_i = 1$
- Choices in normal nodes cover the whole state space.

CFGs with probabilities

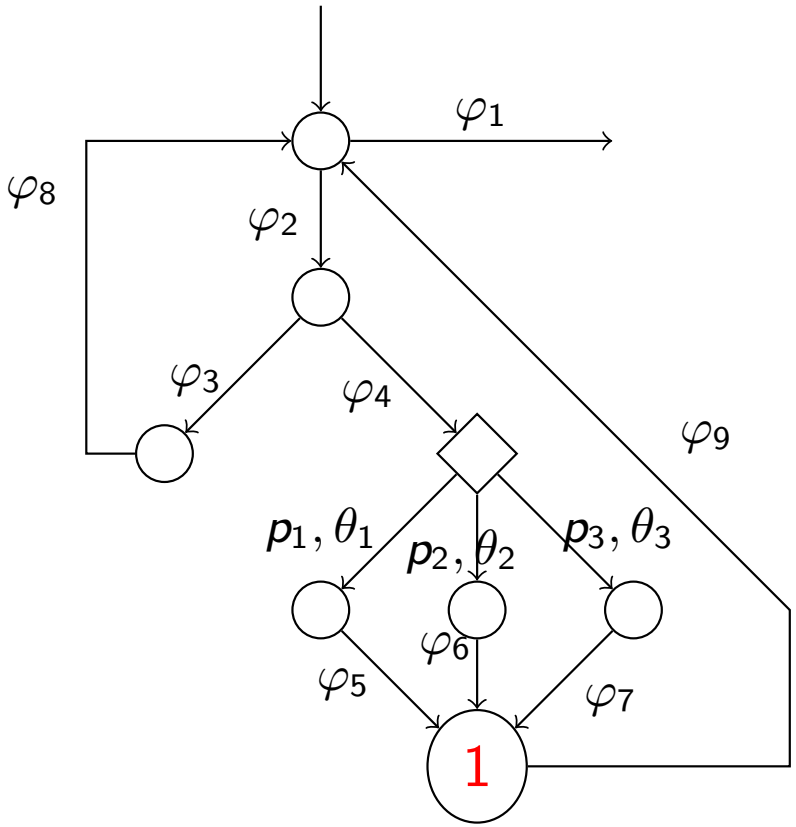
CFG with probabilities:



- Regular instructions represented by circles.
- Probability instructions represented by squares.
- Edges with updates and
  - Probabilities
  - Constraints
- Nodes with cost.
- $\sum_i p_i = 1$
- Choices in normal nodes cover the whole state space.

CFGs with probabilities

CFG with probabilities:



- Regular instructions represented by circles.
- Probability instructions represented by squares.
- Edges with updates and
  - Probabilities
  - Constraints
- Nodes with cost.
- $\sum_i p_i = 1$
- Choices in normal nodes cover the whole state space.

CFGs with probabilities

while $(n > 0)\{$

$n = n - 1;\ \oplus_{\frac{2}{3}}$ skip;

tick(1);

$\}$

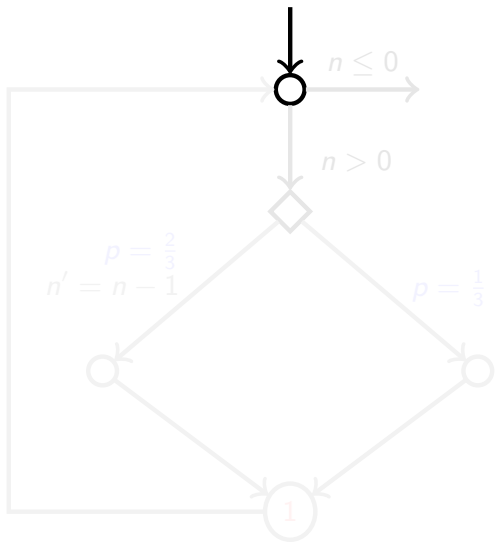
CFGs with probabilities

while $(n > 0)\{$

$n = n - 1;\ \oplus_{\frac{2}{3}}$ skip;

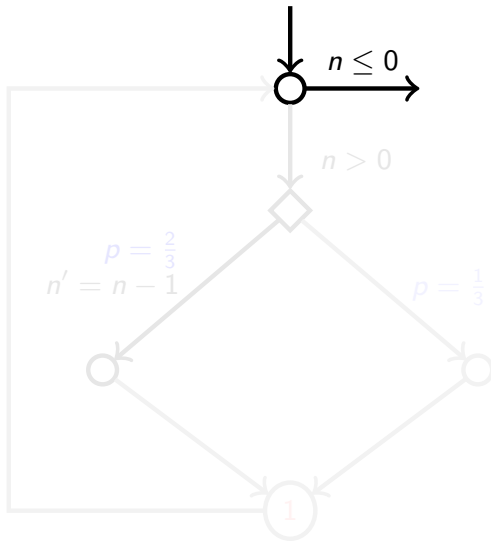
tick(1);

$\}$



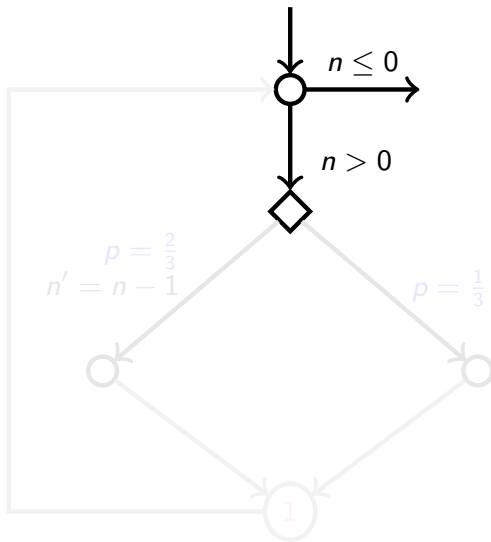
CFGs with probabilities

```
while (n > 0){
    n = n − 1; ⊕_{2/3} skip;
    tick(1);
}
```



CFGs with probabilities

```
while (n > 0){
    n = n − 1; ⊕_{2/3} skip;
    tick(1);
}
```



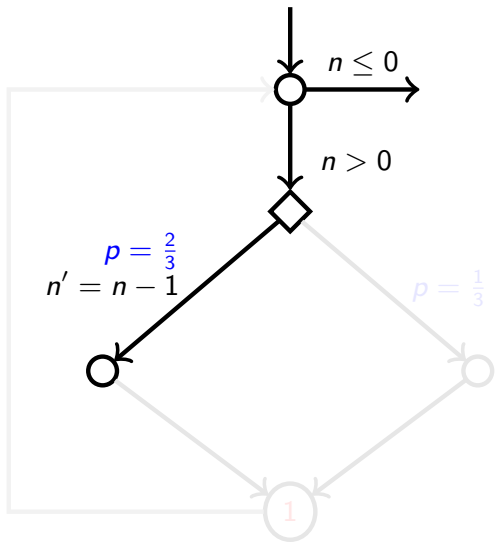
CFGs with probabilities

while $(n > 0)\{$

$n = n - 1;\ \oplus_{\frac{2}{3}}$ skip;

tick(1);

$\}$



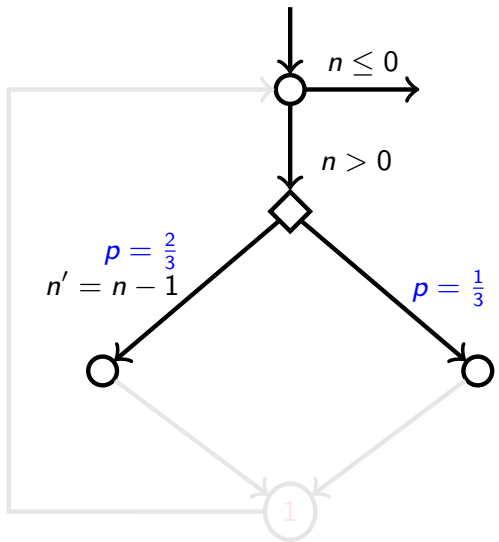
CFGs with probabilities

while $(n > 0)\{$

$n = n - 1;\ \oplus_{\frac{2}{3}}$ skip;

tick(1);

$\}$



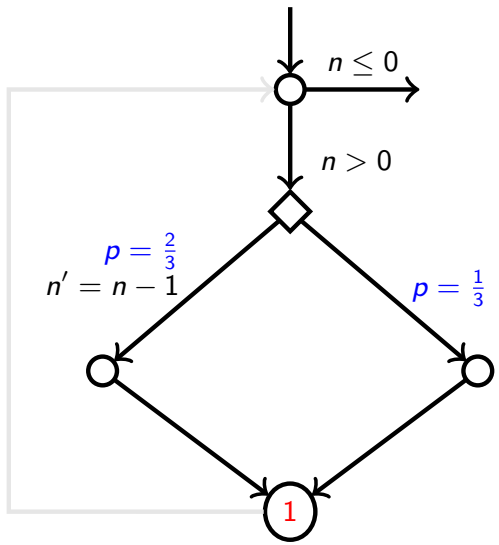
CFGs with probabilities

while $(n > 0)\{$

$n = n - 1;\ \oplus_{\frac{2}{3}}$ skip;

tick(1);

$\}$



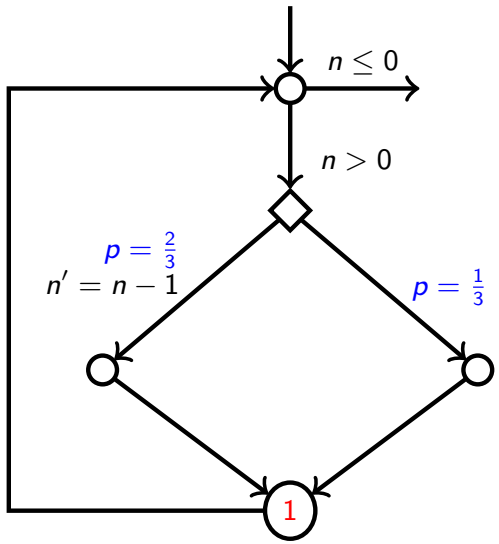
CFGs with probabilities

while $(n > 0)\{$

$n = n - 1;\ \oplus_{\frac{2}{3}}$ skip;

tick(1);

$\}$



Generating CRs

- Each regular node $n \in N_r$ with outgoing transitions $n \xrightarrow{\varphi_i} n_i$, $1 \leq i \leq k$, contributes k cost relations of the form

$$C_n(\bar{x}) = \mathtt{cost}(n) + C_{n_i}(\bar{x}') \qquad \varphi_i$$

- Each probabilistic node $n \in N_p$ with outgoing edges $n \xrightarrow{p_i, \theta_i} n_i$, for $1 \leq i \leq k$, contributes one cost relation

$$C_n(\bar{x}) = \mathtt{cost}(n) + \sum_{i=1}^{k} p_i \cdot C_{n_i}(\bar{x}'_i) \qquad \biguplus_{i=1}^{k} \theta_i$$

Generating CRs

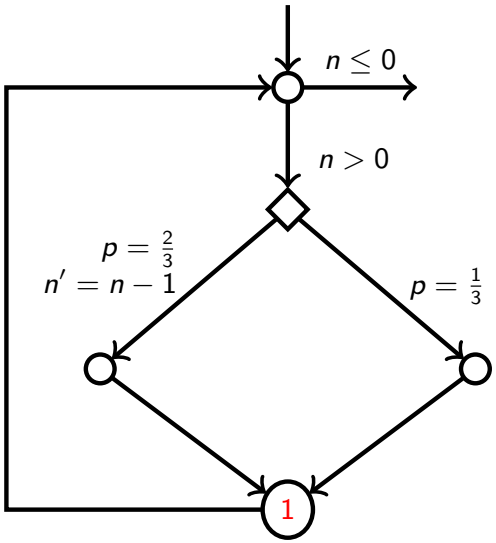
- Each regular node $n \in N_r$ with outgoing transitions $n \xrightarrow{\varphi_i} n_i$, $1 \leq i \leq k$, contributes k cost relations of the form

$$C_n(\bar{x}) = \mathtt{cost}(n) + C_{n_i}(\bar{x}') \qquad \varphi_i$$

- Each probabilistic node $n \in N_p$ with outgoing edges $n \xrightarrow{p_i,\theta_i} n_i$, for $1 \leq i \leq k$, contributes one cost relation

$$C_n(\bar{x}) = \mathtt{cost}(n) + \sum_{i=1}^{k} p_i \cdot C_{n_i}(\bar{x}'_i) \qquad \biguplus_{i=1}^{k} \theta_i$$

Generating CRs

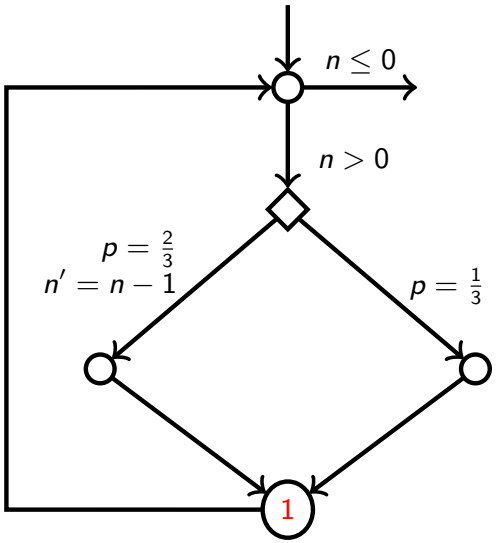


$$
\begin{array}{ll}
C(n) = 0 & \{n \leq 0\} \\
C(n) = 1 + \frac{2}{3}D(n') + \frac{1}{3}D(n'') & \left\{n > 0,\ n' = n - 1,\ n'' = n\right\} \\
D(n) = C(n') & \left\{n' = n\right\}
\end{array}
$$

$$\Downarrow$$

$$
\begin{array}{ll}
C(n) = 0 & \{n \leq 0\} \\
C(n) = 1 + \frac{2}{3}C(n') + \frac{1}{3}C(n'') & \left\{n > 0,\ n' = n - 1,\ n'' = n\right\}
\end{array}
$$

Generating CRs

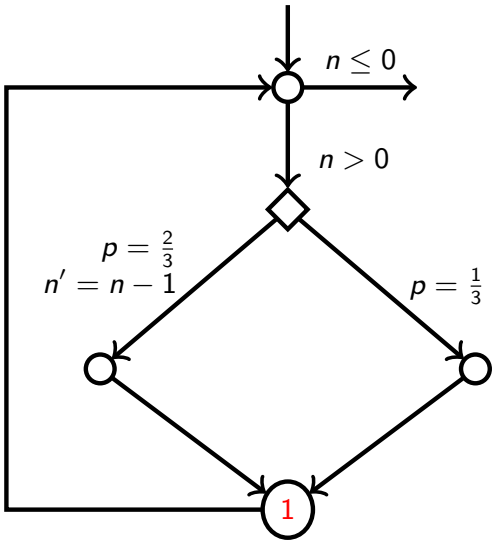


$$
\begin{array}{ll}
C(n) = 0 & \{n \leq 0\} \\
C(n) = 1 + \frac{2}{3}D(n') + \frac{1}{3}D(n'') & \left\{n > 0,\ n' = n - 1,\ n'' = n\right\} \\
D(n) = C(n') & \left\{n' = n\right\}
\end{array}
$$

$\Downarrow$

$$
\begin{array}{ll}
C(n) = 0 & \{n \leq 0\} \\
C(n) = 1 + \frac{2}{3}C(n') + \frac{1}{3}C(n'') & \left\{n > 0,\ n' = n - 1,\ n'' = n\right\}
\end{array}
$$

Generating CRs



$$
\begin{array}{ll}
C(n) = 0 & \{n \leq 0\} \\
C(n) = 1 + \frac{2}{3}D(n') + \frac{1}{3}D(n'') & \left\{n > 0,\ n' = n - 1,\ n'' = n\right\} \\
D(n) = C(n') & \left\{n' = n\right\}
\end{array}
$$

$\Downarrow$

$$
\begin{array}{ll}
C(n) = 0 & \{n \leq 0\} \\
C(n) = 1 + \frac{2}{3}C(n') + \frac{1}{3}C(n'') & \left\{n > 0,\ n' = n - 1,\ n'' = n\right\}
\end{array}
$$

Solving (probabilistic) CRs

$$C(n) = 0 \qquad \underbrace{\{n \leq 0\}}_{\varphi_1}$$

$$C(n) = 1 + \frac{2}{3}C(n') + \frac{1}{3}C(n'') \qquad \underbrace{\{n > 0,\ n' = n-1,\ n'' = n\}}_{\varphi_2}$$

$$\forall n,\ \varphi_1 \implies f(n) \geq 0$$

$$\forall n, n', n'',\ \varphi_2 \implies f(n) \geq 1 + \frac{2}{3}f(n') + \frac{1}{3}f(n'')$$

Solving (probabilistic) CRs

$$C(n) = 0 \qquad \underbrace{\{n \leq 0\}}_{\varphi_1}$$

$$C(n) = 1 + \frac{2}{3}C(n') + \frac{1}{3}C(n'') \qquad \underbrace{\{n > 0,\ n' = n-1,\ n'' = n\}}_{\varphi_2}$$

$$\forall n,\ \varphi_1 \implies f(n) \geq 0$$

$$\forall n, n', n'',\ \varphi_2 \implies f(n) \geq 1 + \frac{2}{3}f(n') + \frac{1}{3}f(n'')$$

Solving (probabilistic) CRs

$$\forall n,\ \varphi_1 \implies f(n) \geq 0$$
$$\forall n,n',n'',\ \varphi_2 \implies f(n) \geq 1 + \frac{2}{3} f(n') + \frac{1}{3} f(n'')$$

$$f(n) = a_1 n + a_0$$
$$\downarrow$$

$$\forall n,\ \varphi_1 \implies a_1 n + a_0 \geq 0$$
$$\forall n,n',n'',\ \varphi_2 \implies a_1 n + a_0 \geq 1 + \frac{2}{3}\left(a_1 n' + a_0\right) + \frac{1}{3}\left(a_1 n + a_0\right)$$

- We use Farkas Lemma to get constraints in terms of a_i from these equations.
- Solving these equations will give us the values for a_i.

Solving (probabilistic) CRs

$$\forall n,\ \varphi_1 \implies f(n) \geq 0$$
$$\forall n,n',n'',\ \varphi_2 \implies f(n) \geq 1 + \frac{2}{3} f(n') + \frac{1}{3} f(n'')$$

$$f(n) = a_1 n + a_0$$
$$\downarrow$$

$$\forall n,\ \varphi_1 \implies a_1 n + a_0 \geq 0$$
$$\forall n,n',n'',\ \varphi_2 \implies a_1 n + a_0 \geq 1 + \frac{2}{3}\left(a_1 n' + a_0\right) + \frac{1}{3}\left(a_1 n + a_0\right)$$

- We use Farkas Lemma to get constraints in terms of a_i from these equations.
- Solving these equations will give us the values for a_i.

Solving (probabilistic) CRs

$$\forall n,\ \varphi_1 \implies f(n) \geq 0$$
$$\forall n,n',n'',\ \varphi_2 \implies f(n) \geq 1 + \frac{2}{3} f(n') + \frac{1}{3} f(n'')$$

$$f(n) = a_1 n + a_0$$
$$\downarrow$$

$$\forall n,\ \varphi_1 \implies a_1 n + a_0 \geq 0$$
$$\forall n,n',n'',\ \varphi_2 \implies a_1 n + a_0 \geq 1 + \frac{2}{3}\left(a_1 n' + a_0\right) + \frac{1}{3}\left(a_1 n + a_0\right)$$

- We use Farkas Lemma to get constraints in terms of a_i from these equations.
- Solving these equations will give us the values for a_i.

Solving (probabilistic) CRs

$$\forall n,\ \varphi_1 \implies f(n) \geq 0$$
$$\forall n,n',n'',\ \varphi_2 \implies f(n) \geq 1 + \frac{2}{3} f(n') + \frac{1}{3} f(n'')$$

$$f(n) = a_1 n + a_0$$
$$\downarrow$$

$$\forall n,\ \varphi_1 \implies a_1 n + a_0 \geq 0$$
$$\forall n,n',n'',\ \varphi_2 \implies a_1 n + a_0 \geq 1 + \frac{2}{3}\left(a_1 n' + a_0\right) + \frac{1}{3}\left(a_1 n + a_0\right)$$

- We use Farkas Lemma to get constraints in terms of a_i from these equations.
- Solving these equations will give us the values for a_i.

Solving (probabilistic) CRs

$$\forall n,\ \varphi_1 \implies f(n) \geq 0$$
$$\forall n,n',n'',\ \varphi_2 \implies f(n) \geq 1 + \frac{2}{3} f(n') + \frac{1}{3} f(n'')$$

$$f(n) = a_1 n + a_0$$
$$\downarrow$$

$$\forall n,\ \varphi_1 \implies a_1 n + a_0 \geq 0$$
$$\forall n,n',n'',\ \varphi_2 \implies a_1 n + a_0 \geq 1 + \frac{2}{3}\left(a_1 n' + a_0\right) + \frac{1}{3}\left(a_1 n + a_0\right)$$

- We use Farkas Lemma to get constraints in terms of a_i from these equations.
- Solving these equations will give us the values for a_i.

Solving (probabilistic) CRs

Upper bound functions must be non-negative.

- We introduce *max* function.
- If we denote

$$F(n) = \max(0, a_1 n + a_0)$$

then we will have

$$\begin{aligned} \forall n,\ \varphi_1 &\implies F(n) \geq 0 \\ \forall n, n', n'',\ \varphi_2 &\implies F(n) \geq 1 + \frac{2}{3}F(n') + \frac{1}{3}F(n) \end{aligned}$$

- Since we are covering the whole state space with the constraints, we have to introduce *max* on both sides of the constraints.

Solving (probabilistic) CRs

Upper bound functions must be non-negative.

- We introduce *max* function.
- If we denote

$$F(n) = \max(0, a_1 n + a_0)$$

then we will have

$$\forall n,\ \varphi_1 \implies F(n) \geq 0$$

$$\forall n, n', n'',\ \varphi_2 \implies F(n) \geq 1 + \frac{2}{3}F(n') + \frac{1}{3}F(n)$$

- Since we are covering the whole state space with the constraints, we have to introduce *max* on both sides of the constraints.

Solving (probabilistic) CRs

$$\forall n,\ \varphi_1,\ a_1 n + a_0 < 0 \implies 0 \geq 0$$
$$\forall n,\ \varphi_1,\ a_1 n + a_0 \geq 0 \implies a_1 n + a_0 \geq 0$$

$$\forall n, n'\ \varphi_2,\ a_1 n + a_0 < 0,\ a_1 n' + a_0 \geq 0 \implies 0 \geq 1 + \frac{2}{3}(a_1 n' + a_0)$$
$$\forall n, n'\ \varphi_2,\ a_1 n + a_0 \geq 0,\ a_1 n' + a_0 < 0 \implies a_1 n + a_0 \geq 1 + \frac{1}{3}(a_1 n + a_0)$$
$$\forall n, n'\ \varphi_2,\ a_1 n + a_0 \geq 0,\ a_1 n' + a_0 \geq 0 \implies a_1 n + a_0 \geq 1 + \frac{2}{3}(a_1 n' + a_0) + \frac{1}{3}(a_1 n + a_0)$$

- Contraints on the left side of the implication lead us to non-linearity.
- We are in trouble with applying Farkas Lemma.

Solving (probabilistic) CRs

$$\forall n,\ \varphi_1,\ a_1 n + a_0 < 0 \implies 0 \geq 0$$
$$\forall n,\ \varphi_1,\ a_1 n + a_0 \geq 0 \implies a_1 n + a_0 \geq 0$$

$$\forall n, n'\ \varphi_2,\ a_1 n + a_0 < 0,\ a_1 n' + a_0 \geq 0 \implies 0 \geq 1 + \frac{2}{3}(a_1 n' + a_0)$$
$$\forall n, n'\ \varphi_2,\ a_1 n + a_0 \geq 0,\ a_1 n' + a_0 < 0 \implies a_1 n + a_0 \geq 1 + \frac{1}{3}(a_1 n + a_0)$$
$$\forall n, n'\ \varphi_2,\ a_1 n + a_0 \geq 0,\ a_1 n' + a_0 \geq 0 \implies a_1 n + a_0 \geq 1 + \frac{2}{3}(a_1 n' + a_0) + \frac{1}{3}(a_1 n + a_0)$$

- Contraints on the left side of the implication lead us to non-linearity.
- We are in trouble with applying Farkas Lemma.

Solving (probabilistic) CRs

$$\forall n,\ \varphi_1,\ a_1 n + a_0 < 0 \implies 0 \geq 0$$
$$\forall n,\ \varphi_1,\ a_1 n + a_0 \geq 0 \implies a_1 n + a_0 \geq 0$$

$$\forall n, n'\ \varphi_2,\ a_1 n + a_0 < 0,\ a_1 n' + a_0 \geq 0 \implies 0 \geq 1 + \frac{2}{3}(a_1 n' + a_0)$$
$$\forall n, n'\ \varphi_2,\ a_1 n + a_0 \geq 0,\ a_1 n' + a_0 < 0 \implies a_1 n + a_0 \geq 1 + \frac{1}{3}(a_1 n + a_0)$$
$$\forall n, n'\ \varphi_2,\ a_1 n + a_0 \geq 0,\ a_1 n' + a_0 \geq 0 \implies a_1 n + a_0 \geq 1 + \frac{2}{3}(a_1 n' + a_0) + \frac{1}{3}(a_1 n + a_0)$$

- Contraints on the left side of the implication lead us to non-linearity.
- We are in trouble with applying Farkas Lemma.

Example: infinite expected cost

while $(n > 0)\{$

$n = n - 1; \ \oplus_{\frac{1}{2}} \ n = n + 1;$

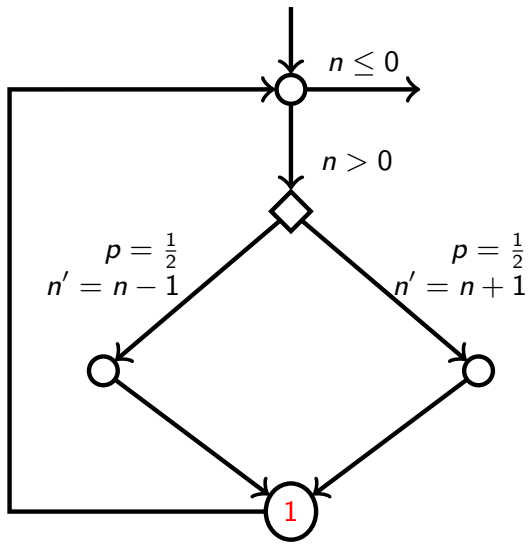
$\}$

Example: infinite expected cost

while $(n > 0)\{$

$n = n - 1;\ \oplus_{\frac{1}{2}}\ n = n + 1;$

$\}$



Multiphase example

while $(x + 3 \leq n)\{$

  if $(y < m)$

    skip; $\oplus_{\frac{1}{2}} y = y + 1;$

  else

    skip; $\oplus_{\frac{1}{4}} x = x + 1; \oplus_{\frac{1}{4}} x = x + 2; \oplus_{\frac{1}{4}} x = x + 3;$

  tick(1);

$\}$

$$f(x, y, n, m) = \max(0, a_1 x + a_2 y + a_3 n + a_4 m + a_0)$$

If we use this global template... we will fail.

Multiphase example

while $(x + 3 \leq n)\{$

  if $(y < m)$

    skip; $\oplus_{\frac{1}{2}} y = y + 1;$

  else

    skip; $\oplus_{\frac{1}{4}} x = x + 1; \oplus_{\frac{1}{4}} x = x + 2; \oplus_{\frac{1}{4}} x = x + 3;$

  tick(1);

$\}$

$$f(x, y, n, m) = \max(0, a_1 x + a_2 y + a_3 n + a_4 m + a_0)$$

If we use this global template... we will fail.

Multiphase example

while $(x + 3 \leq n)\{$

  if $(y < m)$

    skip; $\oplus_{\frac{1}{2}} y = y + 1;$

  else

    skip; $\oplus_{\frac{1}{4}} x = x + 1; \oplus_{\frac{1}{4}} x = x + 2; \oplus_{\frac{1}{4}} x = x + 3;$

  tick(1);

$\}$

$$f(x, y, n, m) = \max(0, a_1 x + a_2 y + a_3 n + a_4 m + a_0)$$

If we use this global template... we will fail.

Multiphase example

```
while (x + 3 ≤ n){
    if (y < m)
        skip; ⊕_{1/2} y = y + 1;
    else
        skip; ⊕_{1/4} x = x + 1; ⊕_{1/4} x = x + 2; ⊕_{1/4} x = x + 3;
    tick(1);
}
```

We can start from a template that includes two parts (one for each phase).

$$f(x, y, n, m) = \max(0, a_1 x + a_2 y + a_3 n + a_4 m + a_0) + \max(0, b_1 x + b_2 y + b_3 n + b_4 m + b_0)$$

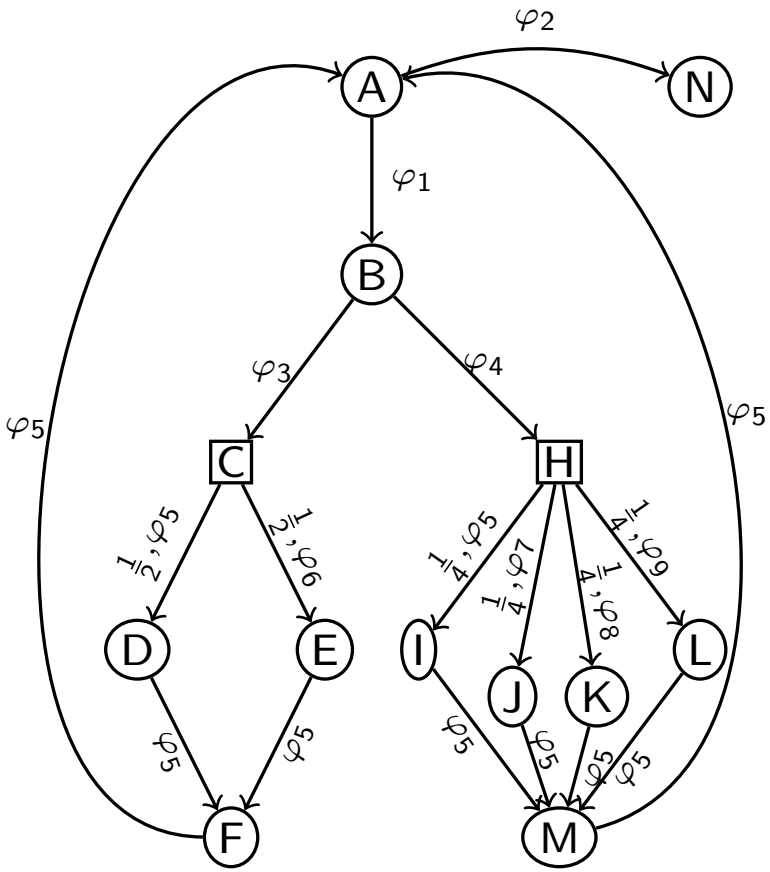
Multiphase example

while $(x + 3 \leq n)\{$
  if $(y < m)$
    skip; $\oplus_{\frac{1}{2}} y = y + 1$;
  else
    skip; $\oplus_{\frac{1}{4}} x = x + 1$; $\oplus_{\frac{1}{4}} x = x + 2$; $\oplus_{\frac{1}{4}} x = x + 3$;
  tick(1);
$\}$

We can start from a template that includes two parts (one for each phase).

$$f(x, y, n, m) = \max(0, a_1 x + a_2 y + a_3 n + a_4 m + a_0) + \max(0, b_1 x + b_2 y + b_3 n + b_4 m + b_0)$$

Multiphase example



$$
\begin{aligned}
\varphi_1 &= \{x+3 \leq n\} \cup \varphi_5 \\
\varphi_2 &= \{x+3 > n\} \cup \varphi_5 \\
\varphi_3 &= \{y < m\} \cup \varphi_5 \\
\varphi_4 &= \{y' \geq m\} \cup \varphi_5 \\
\varphi_5 &= \{x' = x, n' = n, y' = y, m' = m\} \\
\varphi_6 &= \{x' = x, n' = n, y' = y+1, m' = m\} \\
\varphi_7 &= \{x' = x+1, n' = n, y' = y, m' = m\} \\
\varphi_8 &= \{x' = x+2, n' = n, y' = y, m' = m\} \\
\varphi_9 &= \{x' = x+3, n' = n, y' = y, m' = m\}
\end{aligned}
$$

Conclusions

- Our interest is to extend SACO to handle probabilistic programs
- We have extended the CFGs to have probabilities
- We have modeled the expected cost of CFGs with (probability) CRs.
- We have developed a technique for solving such CRs.
- Correctness: based on Farkas Lemma and Markov Decision Processes.