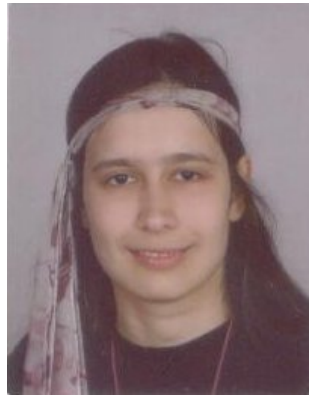
Improving Static Dependency Pairs for Higher-Order Rewriting

Carsten Fuhs



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Radboud Universiteit Nijmegen, The Netherlands

16th Workshop on Termination

Oxford, UK

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Why is termination of *higher-order* rewriting interesting?

Recently:

- Broader interest in termination analysis tools for higher-order rewriting
- Categories at the **Termination Competition**
  - "Higher-order rewriting union β": since 2010
  - "Rewriting with higher-order pattern matching on β-normal η-long forms (HRS)": under discussion
- New tools at TermComp 2017 (`SOL`) and 2018 (`SizeChangeTool`)
- Growing number of benchmarks in the **TPDB**
- Tools also used as back-end oracles in **Confluence Competition**

Higher-order rewriting in practice

- Specification language (with syntactic sugar) for **compiler synthesis**:
  CRSX [Rose, *RTA'11*] (successor tool `TransScript` used at IBM)

  `http://crsx.sourceforge.net/`

  Termination of higher-order specification
  $\Rightarrow$ **termination of the executable compiler**

- Termination analysis for functional programs

- Nuances in definitions of higher-order rewriting matter

  $\rightarrow$ discussion on `termtools` mailing list about
  plain matching vs matching *modulo beta*
  at TermComp 2018

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- allows transformations from many formalisms of interest for termination
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- also covers (second-order) pattern HRSs ($\rightarrow$ CoCo)

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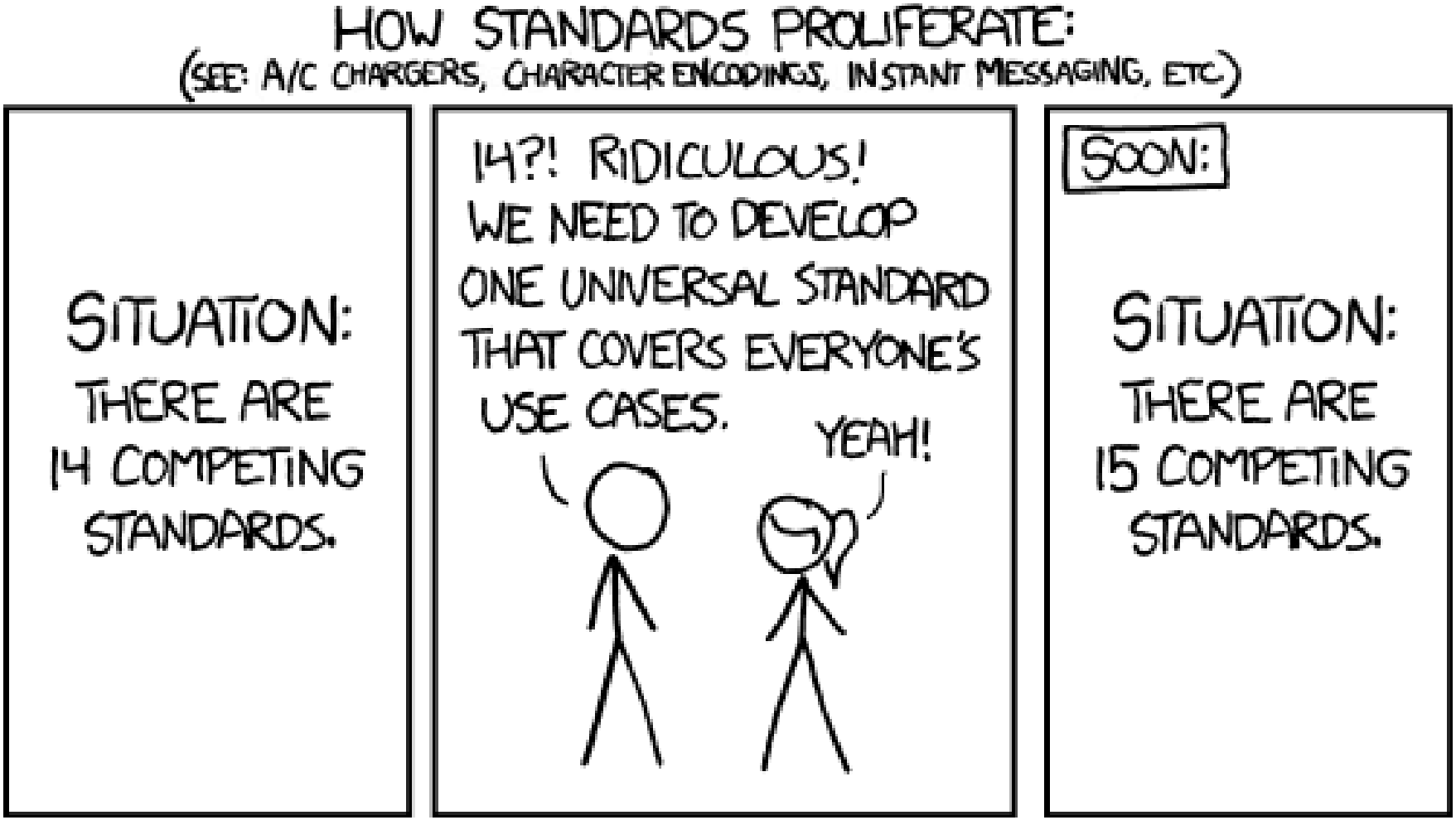

https://xkcd.com/927/

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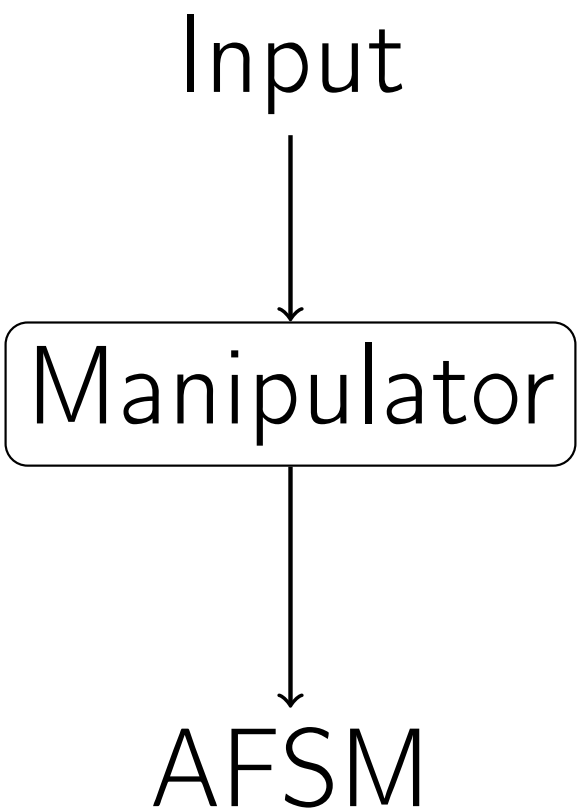
AFSMs in Wanda at TermComp, CoCo, ...



Termination of First-Order Term Rewriting: append

Example (append)

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Dependency Pairs and the Subterm Criterion

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- Extend to terms: $\nu(\ f^\sharp\ t_1\ \ldots\ t_n\ ) = t_{\nu(f^\sharp)}$

  $\nu(\ \mathsf{append}^\sharp\ (\mathsf{cons}\ x\ xs)\ ys\ ) = \mathsf{cons}\ x\ xs$

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Show termination using Dependency Pairs [Arts, Giesl, *TCS'00*]:

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Dependency Pairs and the Subterm Criterion

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$$\begin{array}{rcl} \mathsf{map}\ F\ \mathsf{nil} & \to & \mathsf{nil} \\ \mathsf{map}\ F\ (\mathsf{cons}\ x\ xs) & \to & \mathsf{cons}\ (F\ x)\ (\mathsf{map}\ F\ xs) \end{array}$$

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Example ( )

$$\mathcal{R}: \quad \frac{\mathsf{app}\ (\mathsf{lam}\ F)\ x \quad \to \quad F\ x}{\mathsf{app} : \mathsf{o} \Rightarrow \mathsf{o} \Rightarrow \mathsf{o}, \qquad \mathsf{lam} : (\mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o}}$$

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So can't we just prohibit Static DPs for such malicious inputs?

Plain Function Passing (PFP)

Original soundness criterion for Static DPs [Kusakari, Isogai, Sakai, Blanqui, *TIS'09*] (adapted to our AFSMs):

Definition (Plain Function Passing (PFP))

A rewrite system $\mathcal{R}$ is Plain Function Passing (PFP)
if for all rules $f\ \ell_1\ \ldots\ \ell_n \to r \in \mathcal{R}$ and for all meta-variables F
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Plain Function Passing (PFP)

Original soundness criterion for Static DPs [Kusakari, Isogai, Sakai, Blanqui, *TIS'09*] (adapted to our AFSMs):

Definition (Plain Function Passing (PFP))

A rewrite system $\mathcal{R}$ is Plain Function Passing (PFP)
if for all rules $f\ \ell_1\ \ldots\ \ell_n \to r \in \mathcal{R}$ and for all meta-variables F
of functional type in r, we have $F = \ell_i$ for some i.

Theorem (Static DPs sound for Plain Function Passing $\mathcal{R}$)

Let $\mathcal{R}$ be a rewrite system that is PFP. Then Static DPs are sound for proving termination of $\mathcal{R}$.

Which of the following rules is PFP?

- append $(\text{cons}\ x\ xs)\ ys \to \text{cons}\ x\ (\text{append}\ xs\ ys)$ ✓
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PFP improved in [Suzuki, Kusakari, Blanqui, *IPSJ TOP'11*].

Termination of Higher-Order Term Rewriting: ord-rec

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Idea: decrease in a **well-founded order** on types when we apply an F taken out of a left-hand side

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→ Accessibility, see General Schema [Blanqui, Jouannaud, Okada, *TCS'02*], Computability Path Ordering [Blanqui, Jouannaud, Rubio, *LMCS'15*], and Computability Closure [Blanqui, *TCS'16*]

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Related criterion (incomparable): [Blanqui, *WST'06*]

Static DPs and the Computable Subterm Criterion

Example (Static DPs for ord-rec)

$$\begin{aligned} \mathsf{rec}^\sharp\ (\mathsf{s}\ X)\ K\ F\ G &\Rightarrow \mathsf{rec}^\sharp\ X\ K\ F\ G \\ \mathsf{rec}^\sharp\ (\mathsf{lim}\ H)\ K\ F\ G &\Rightarrow \mathsf{rec}^\sharp\ (H\ M)\ K\ F\ G \end{aligned}$$

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Candidate for Subterm Criterion: $\nu(\mathsf{rec}^\sharp) = 1$.

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Candidate for Subterm Criterion: $\nu(\mathsf{rec}^\sharp) = 1$.

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$\Rightarrow$ ord-rec is terminating by **Computable Subterm Criterion**

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Here:

- Make Static DPs applicable to larger class of higher-order systems
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Further extensions available:

https://arxiv.org/abs/1805.09390

- Unified framework integrating static and dynamic DPs (dynamic DPs include DPs for calls to meta-variables of functional type)
- Many additional termination proving techniques as DP processors
- Completeness results wrt non-termination for static and dynamic DPs

Bonus slide: Plain Function Passing, v2

PFP: Must the F really be direct arguments on left-hand side?

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Example (deriv)

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