

Graduate
Workshop
“CGI”

Quality Ratios
in
Graph Drawing

Marc van Kreveld - Utrecht University

Largely based
on:

Quality Ratios of **Measures** for Graph Drawing Styles

Michael Hoffmann - Zurich
Marc van Kreveld - Utrecht
Vincent Kusters - Zurich
Günter Rote - Berlin

and a little on:

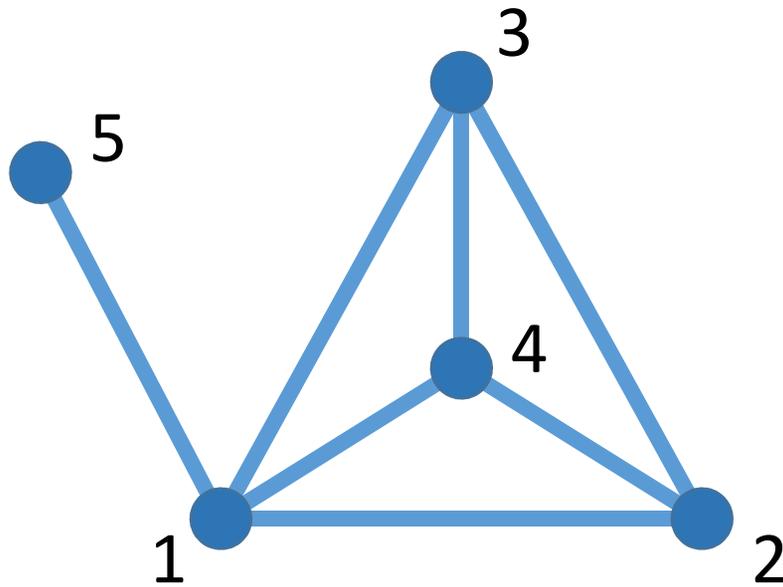
**On the Quality Ratio
of RAC Drawings
and Planar Drawings
for Planar
Graphs**

Marc van Kreveld - Utrecht

$V = \{1,2,3,4,5\}$

$E = \{ (1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (3,4) \}$

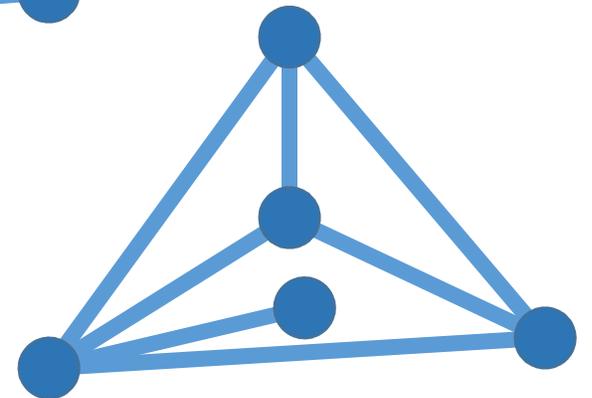
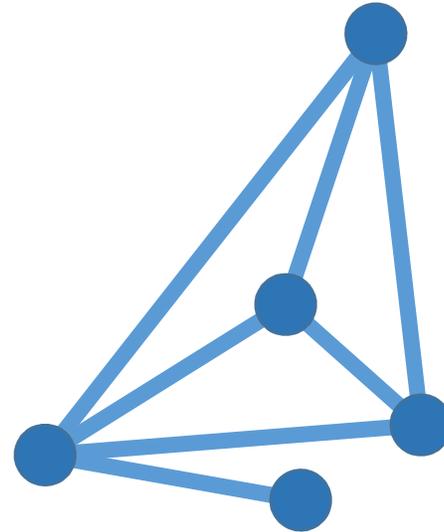
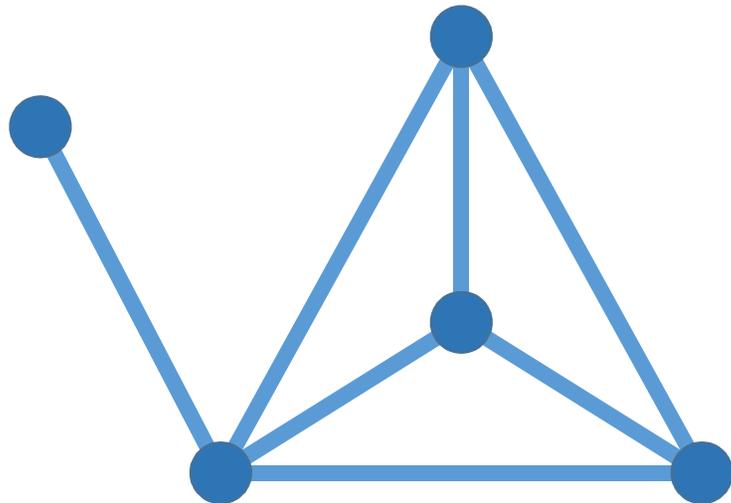
Graph Drawing



$V = \{1,2,3,4,5\}$

$E = \{ (1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (3,4) \}$

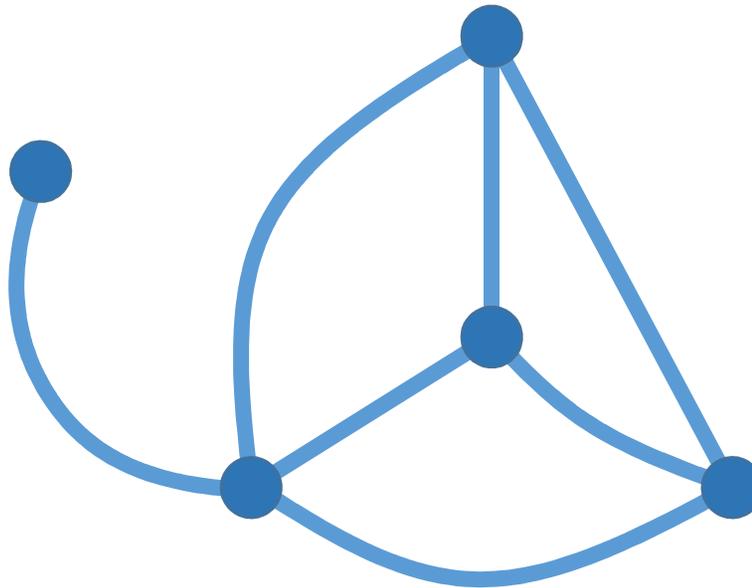
Graph Drawing



$V = \{1,2,3,4,5\}$

$E = \{ (1,2), (1,3),$
 $(1,4), (1,5),$
 $(2,3), (2,4),$
 $(3,4) \}$

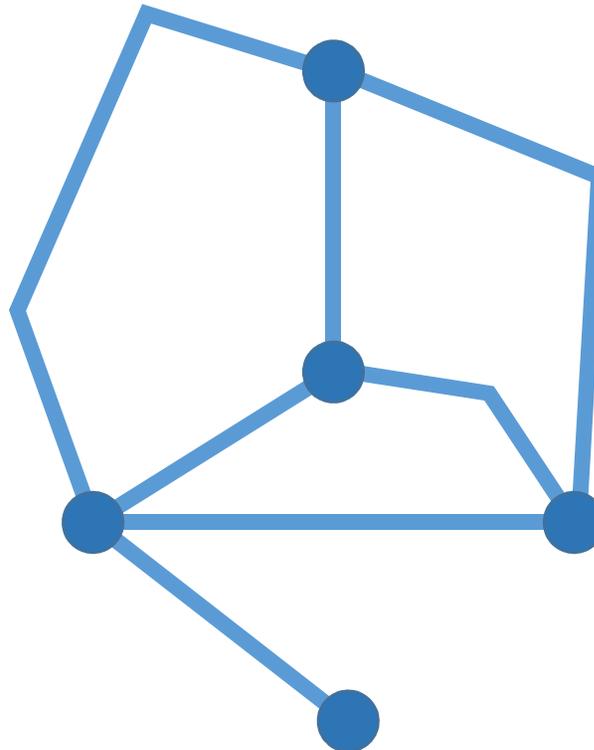
Graph Drawing



$V = \{1,2,3,4,5\}$

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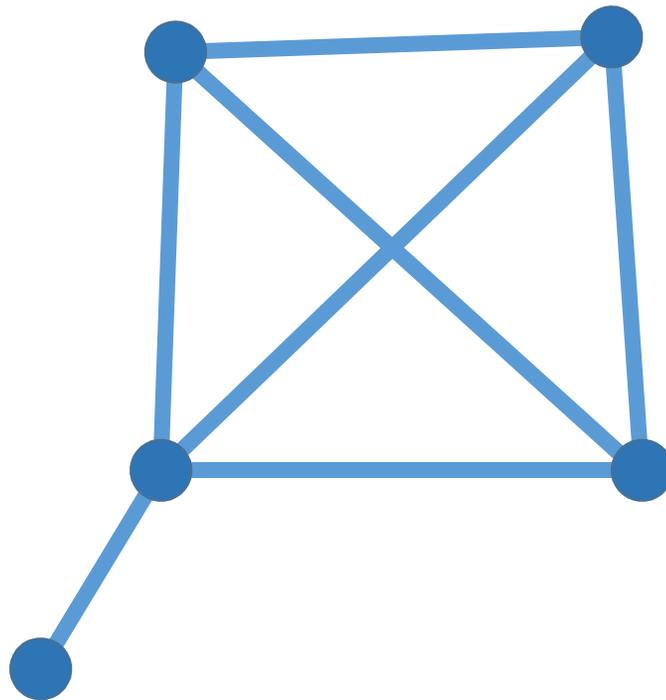
Graph Drawing



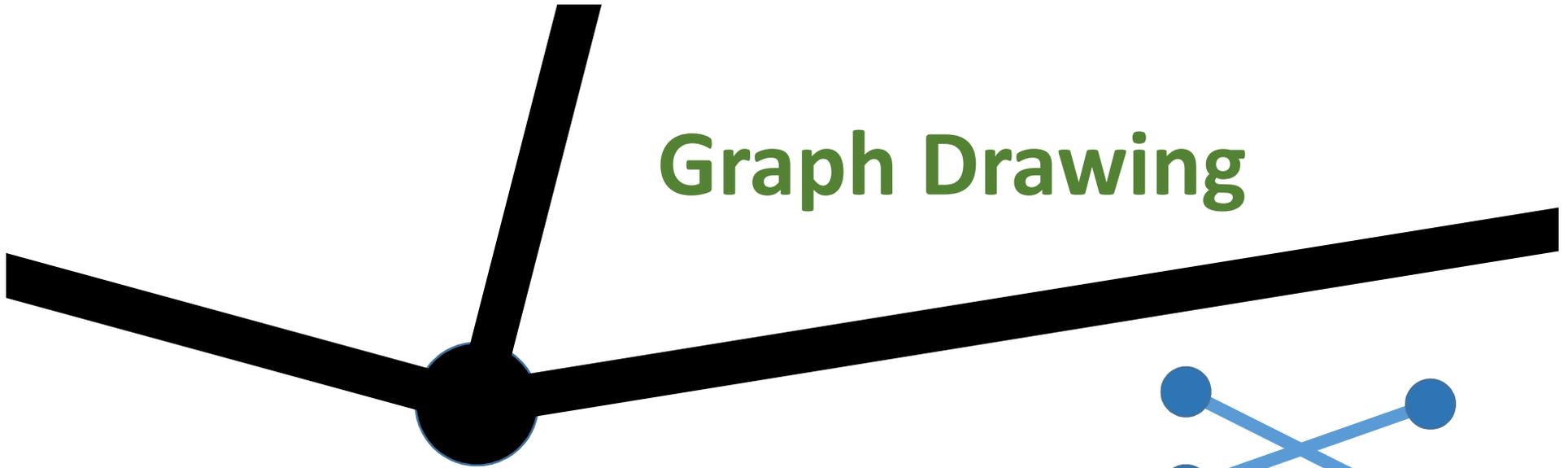
$V = \{1,2,3,4,5\}$

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Graph Drawing

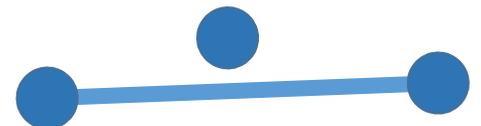
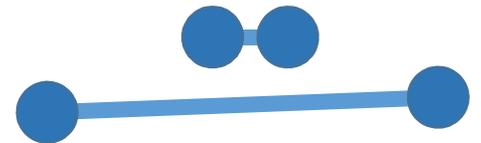
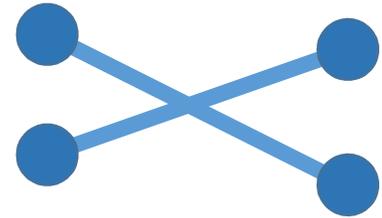


Graph Drawing



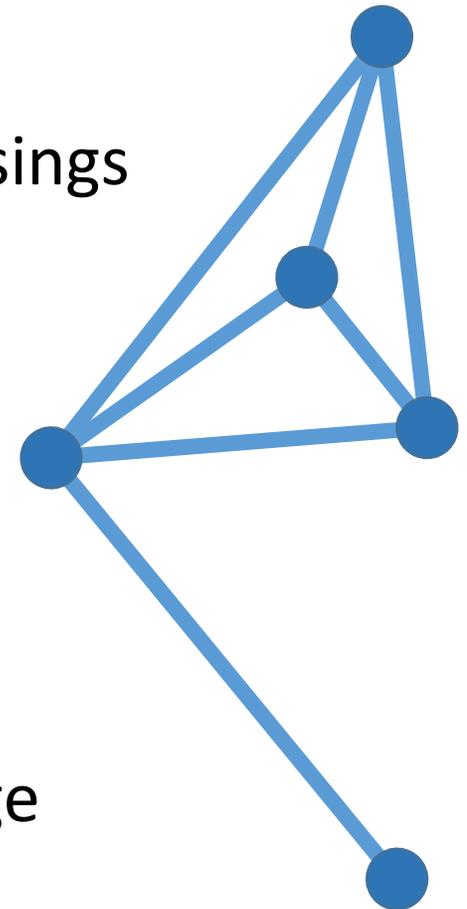
Don't like:

- edge-edge crossings
- small angles between incident edges
- both long and short edges
- edges close to non-incident vertices



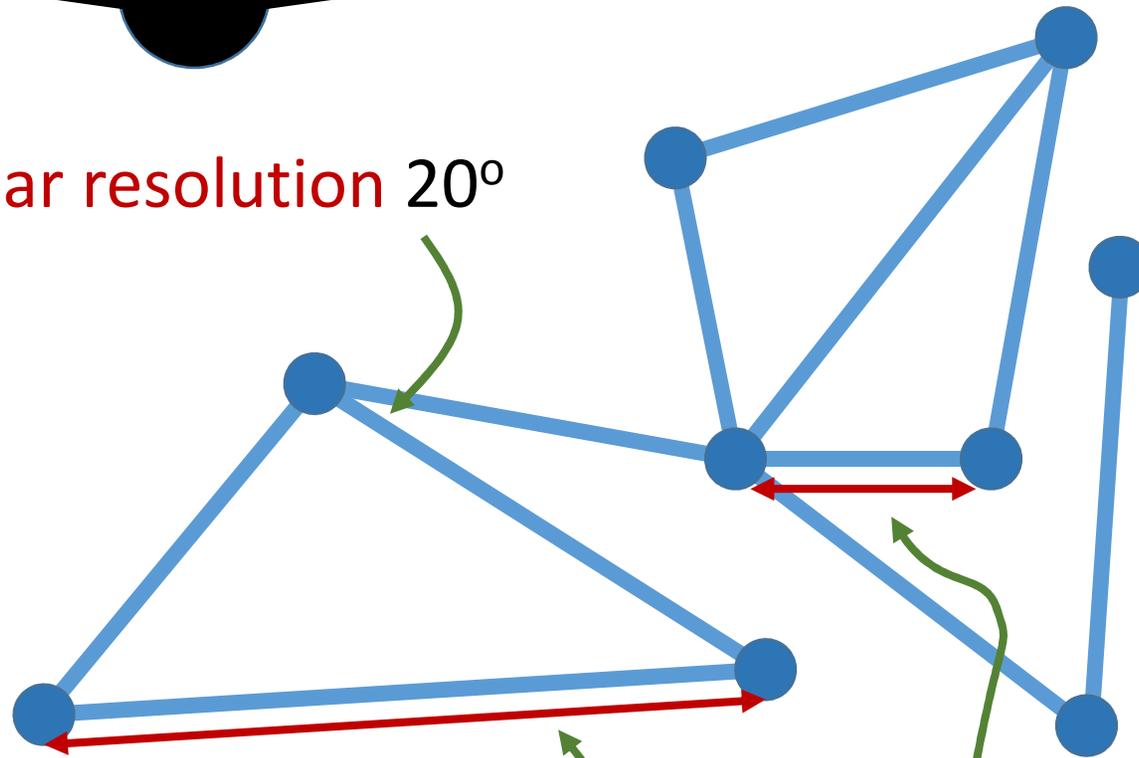
Quality Measures

- **crossing number**: number of edge crossings
- **angular resolution**: smallest angle between incident edges
- **edge-length ratio**: ratio of longest to shortest edge length
- **area requirement**: grid size needed
- **feature resolution** : ratio of longest edge to shortest vertex-edge distance



Quality Measures

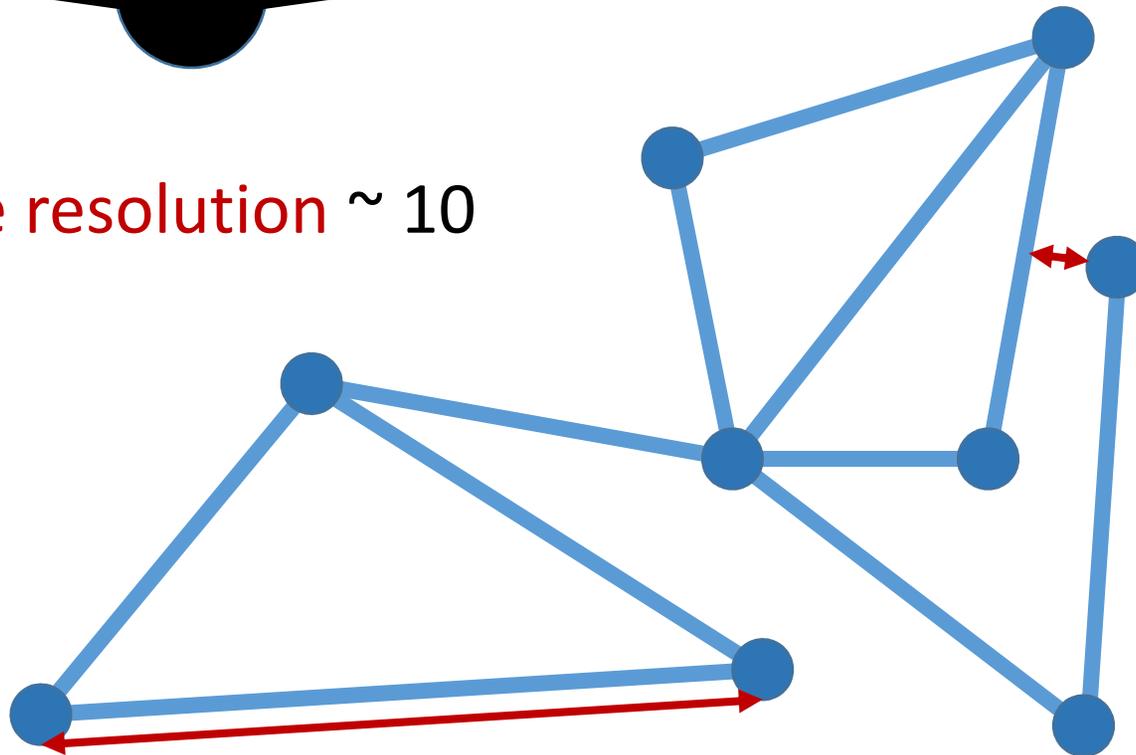
angular resolution 20°



edge-length ratio ~ 3

Quality Measures

feature resolution ~ 10

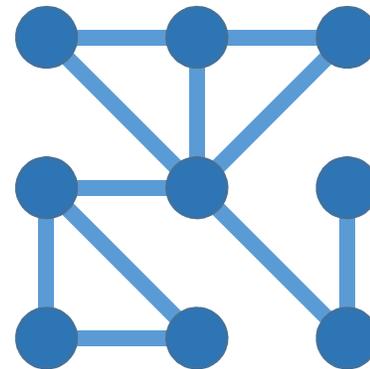


Quality Measures

angular resolution

edge-length ratio

area requirement

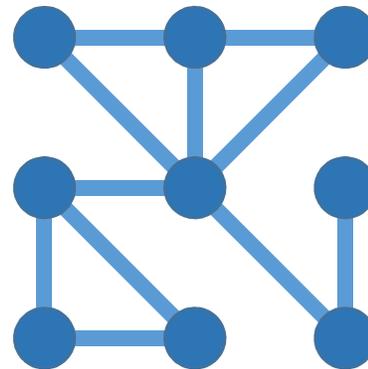


Quality Measures

angular resolution 45°

edge-length ratio $\sqrt{2}$

area requirement 9

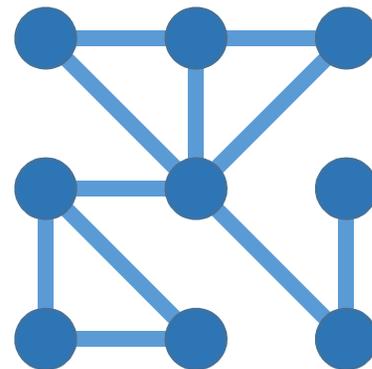
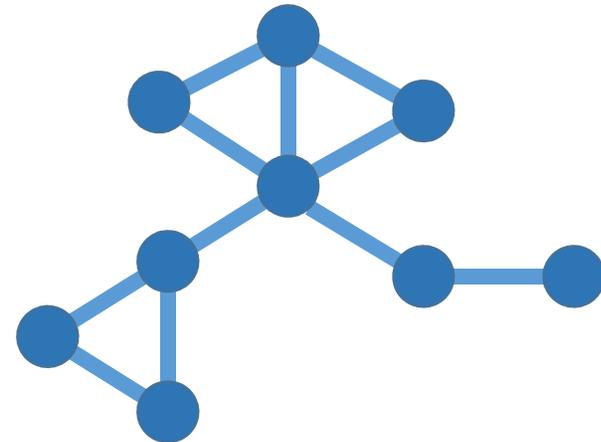


Quality Measures

angular resolution 60°

edge-length ratio 1

area requirement 9



Quality Measures



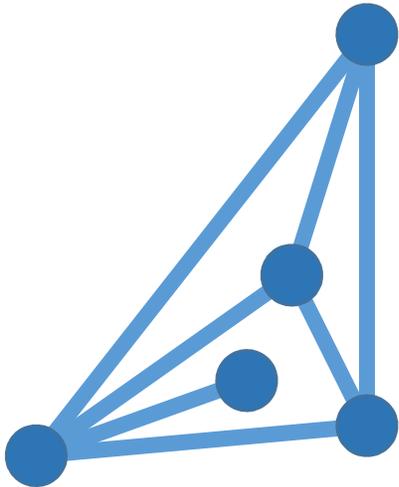
quality measure of a drawing of a graph G

not the same thing as

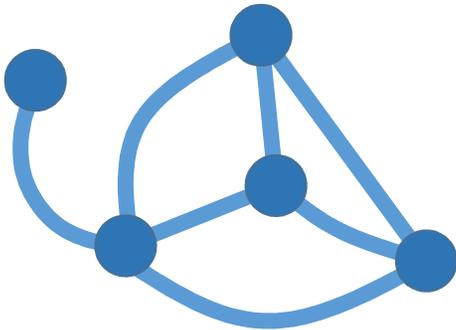
quality measure of a graph G

Drawing Styles

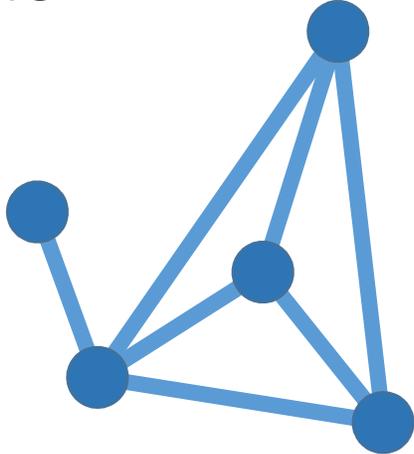
fixed embedding
planar straight line



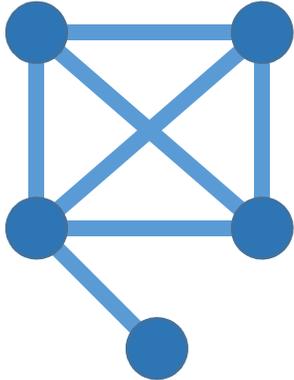
free embedding
planar **circular** arc



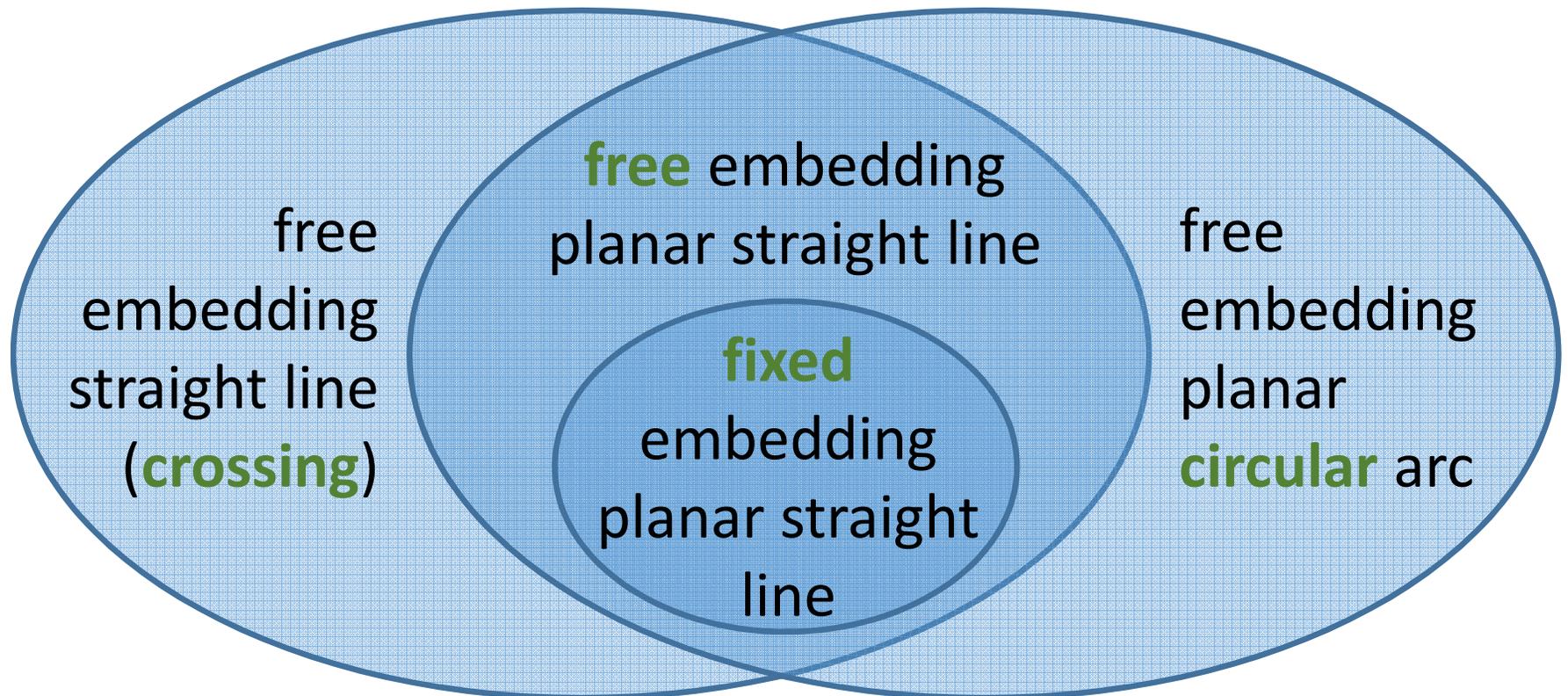
free embedding
planar straight line

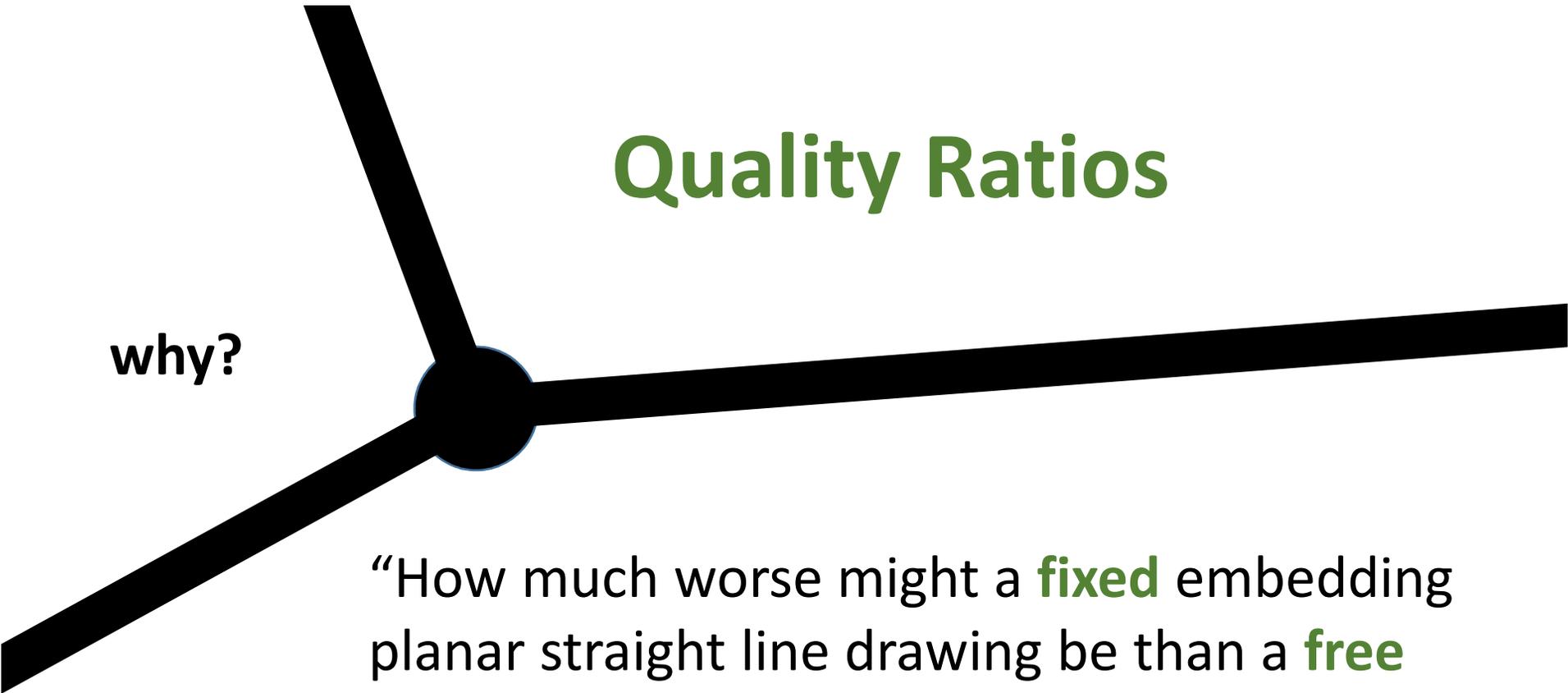


free embedding
straight line (**crossing**)



Drawing Styles





Quality Ratios

why?

“How much worse might a **fixed** embedding planar straight line drawing be than a **free** embedding planar straight line drawing in terms of **angular resolution**?”

“How much better can a **circular** arc drawing of a graph be than a **free** planar straight line drawing in terms of **area requirement**?”

Quality Ratios

definition

for **angular resolution**; **fixed**
versus **free** plane drawings

$$\text{QR}(\text{free}, \text{fixed}) = \sup_{\text{planar graph } G} \frac{\text{angular resolution of free plane drawing of } G}{\text{angular resolution of fixed straight plane drawing of } G}$$

Quality Ratios

definition

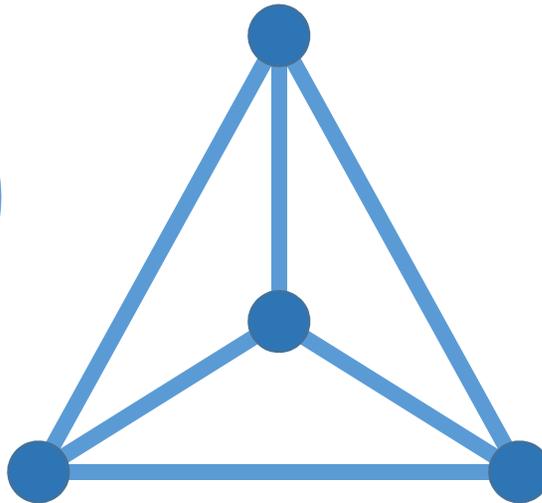
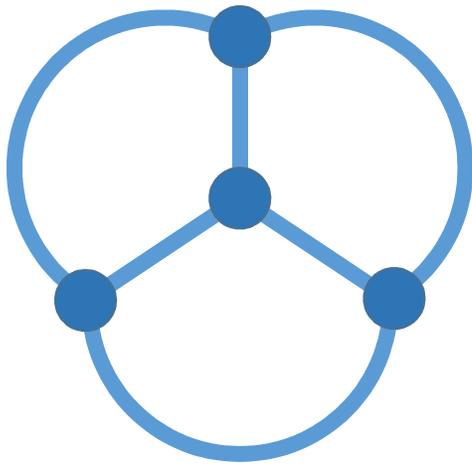
for **area requirement**; **circular**
versus **free** embedding straight
plane drawings

$$\text{QR}(\mathbf{circular}, \mathbf{free}) = \sup_{\substack{\text{planar} \\ \text{graph } G}} \frac{\text{area requirement of} \\ \mathbf{circular} \text{ plane drawing of } G}{\text{area requirement of} \\ \mathbf{free} \text{ plane drawing of } G}$$

Quality Ratios

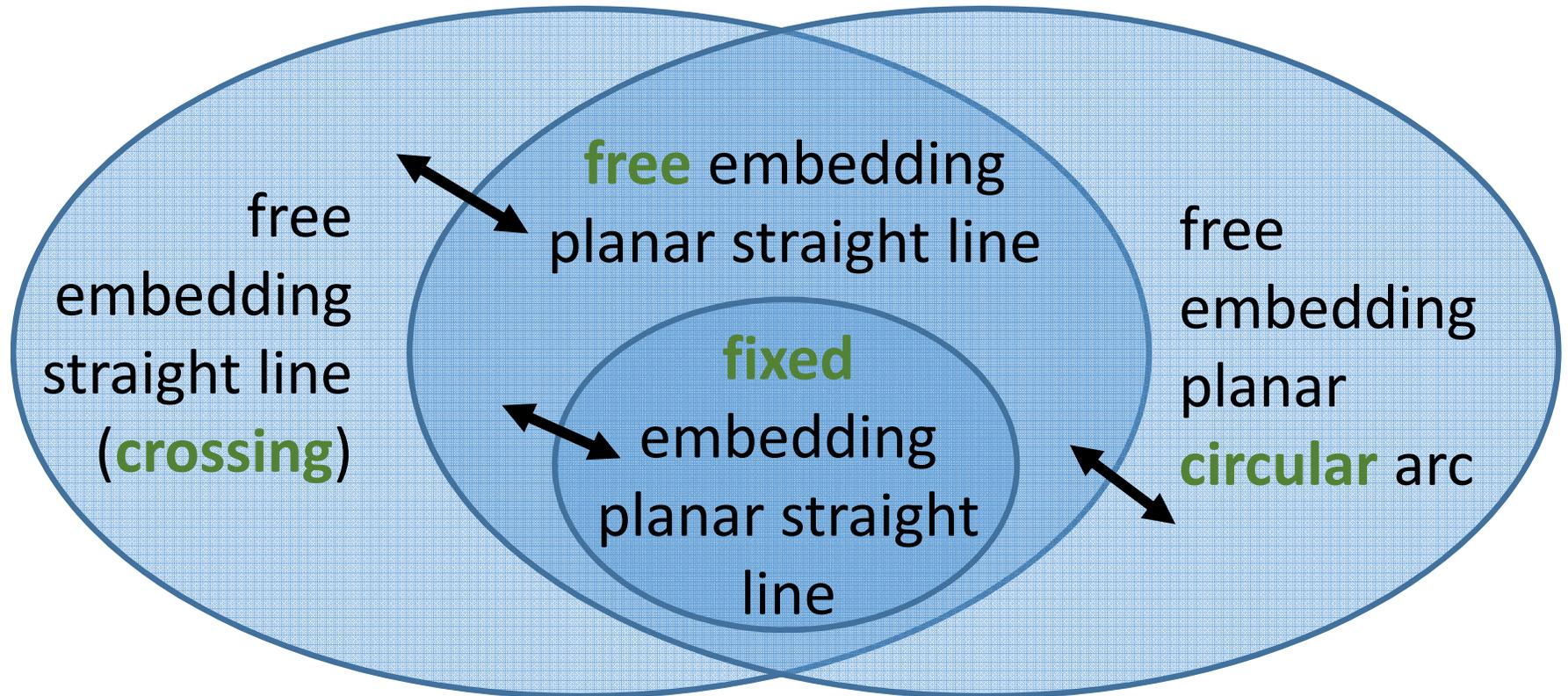
example

Angular resolution; **circular**
versus **free** plane drawings



$$QR \geq 120/30 = 4$$

Quality Ratios



Results

planar graphs

free : fixed

circ. : free

crossing : free

angular resolution

≥ 4.8

area requirement

edge-length ratio

feature resolution

Results

planar graphs

	free : fixed	circ. : free	crossing : free
angular resolution	≥ 12	≥ 4.8	∞
area requirement	∞	∞	∞
edge-length ratio	∞	∞	∞
feature resolution	∞	$\geq 3\sqrt{3} / \pi$	≥ 2.509

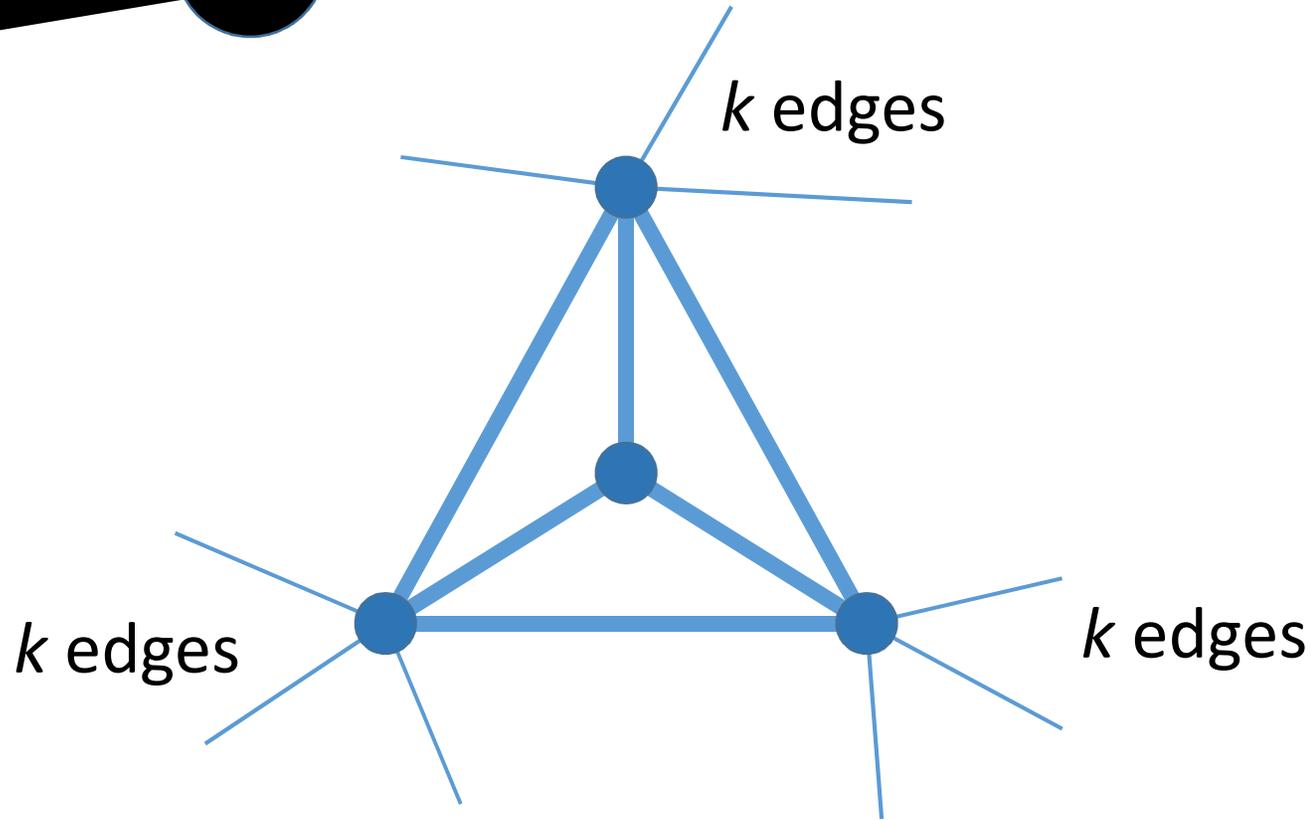
Results

planar graphs

	free : fixed	circ. : free	crossing : free
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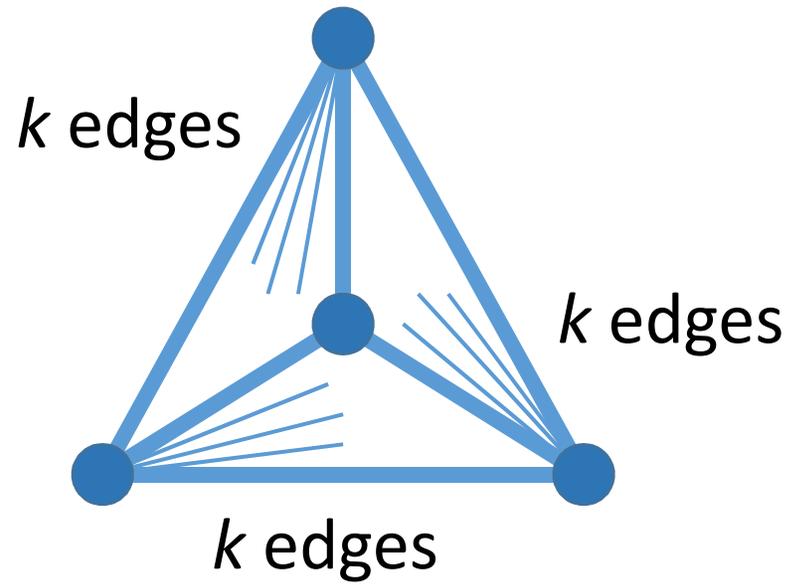
free : fixed

angular
resolution



free : fixed

angular
resolution



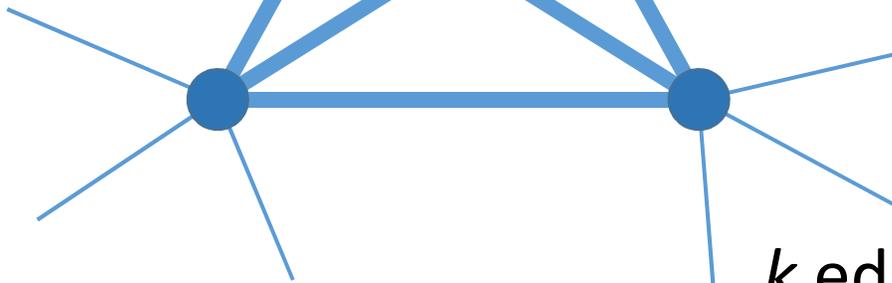
free : fixed

angular
resolution



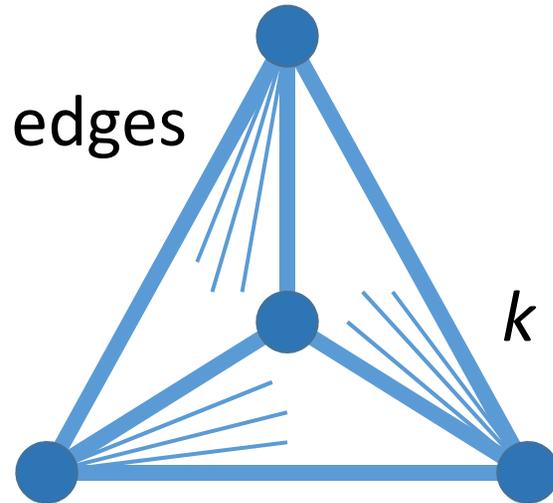
k edges

k edges



k edges

k edges



k edges

k edges

free : fixed

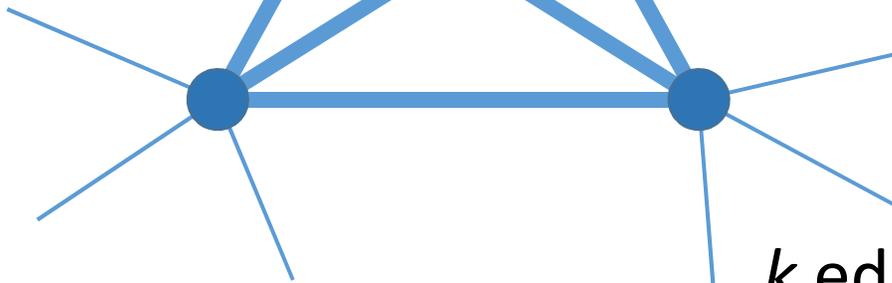
angular
resolution



k edges

$360 / (k+3)$

k edges

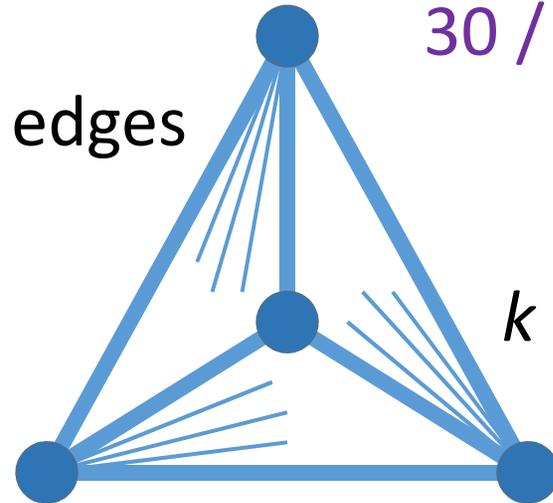


k edges

k edges

$30 / (k+1)$

k edges



k edges

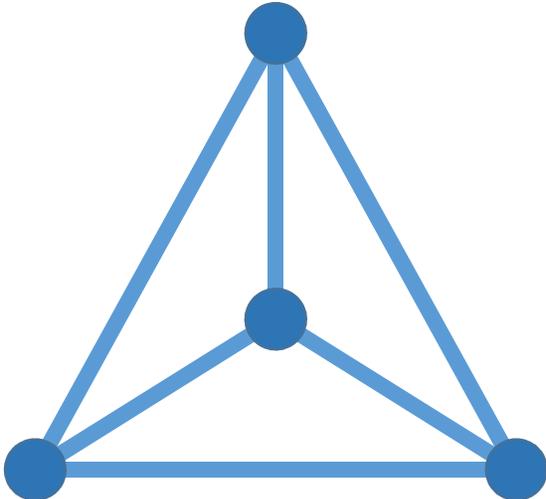
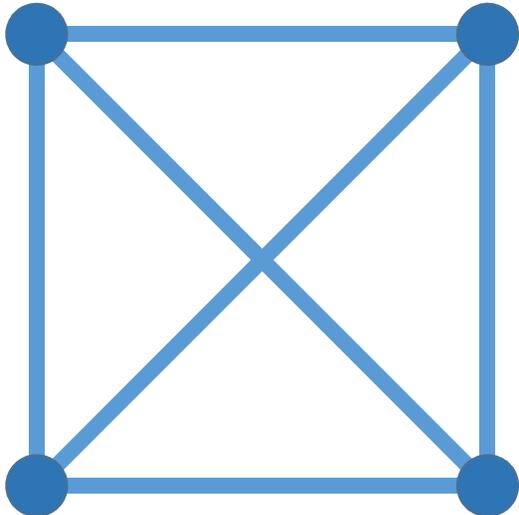
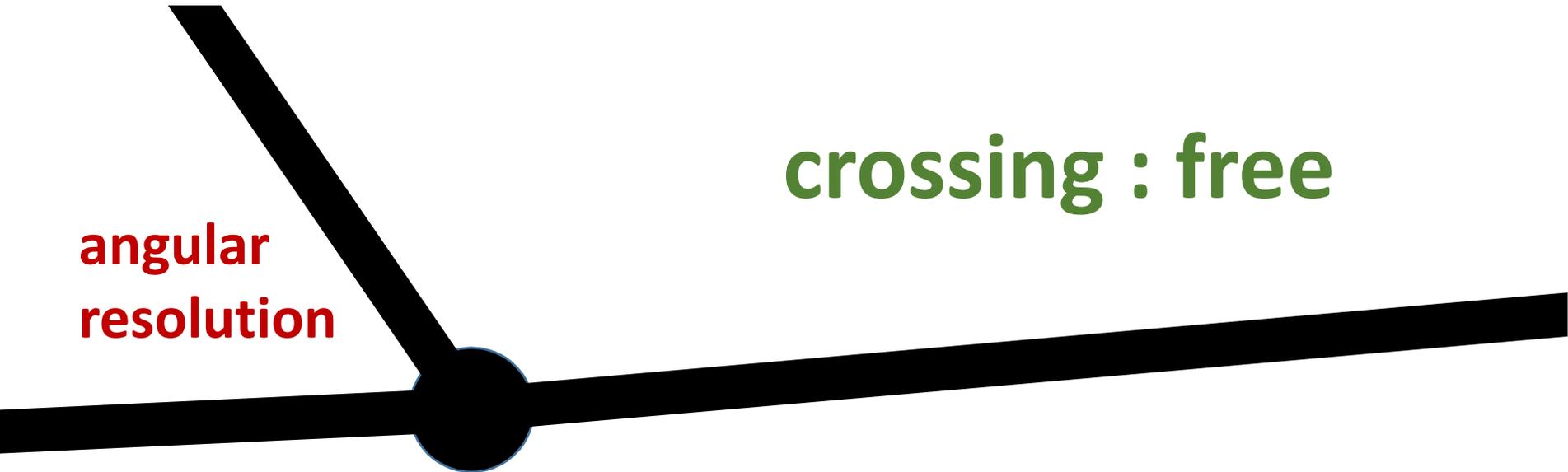


free : fixed

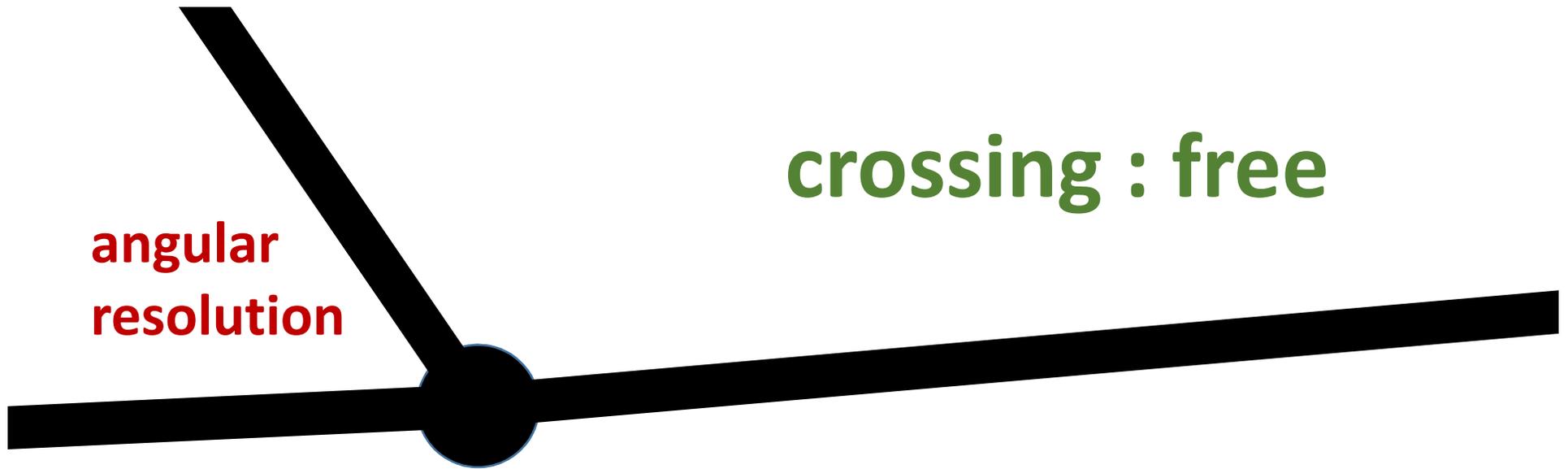
angular
resolution

$$\text{QR}(\text{free}, \text{fixed}) = \sup_{\text{planar graph } G} \frac{\text{angular resolution of free plane drawing of } G}{\text{angular resolution of fixed plane drawing of } G}$$

$$\geq \sup_k \frac{360 / (k+3)}{30 / (k+1)} = 12$$

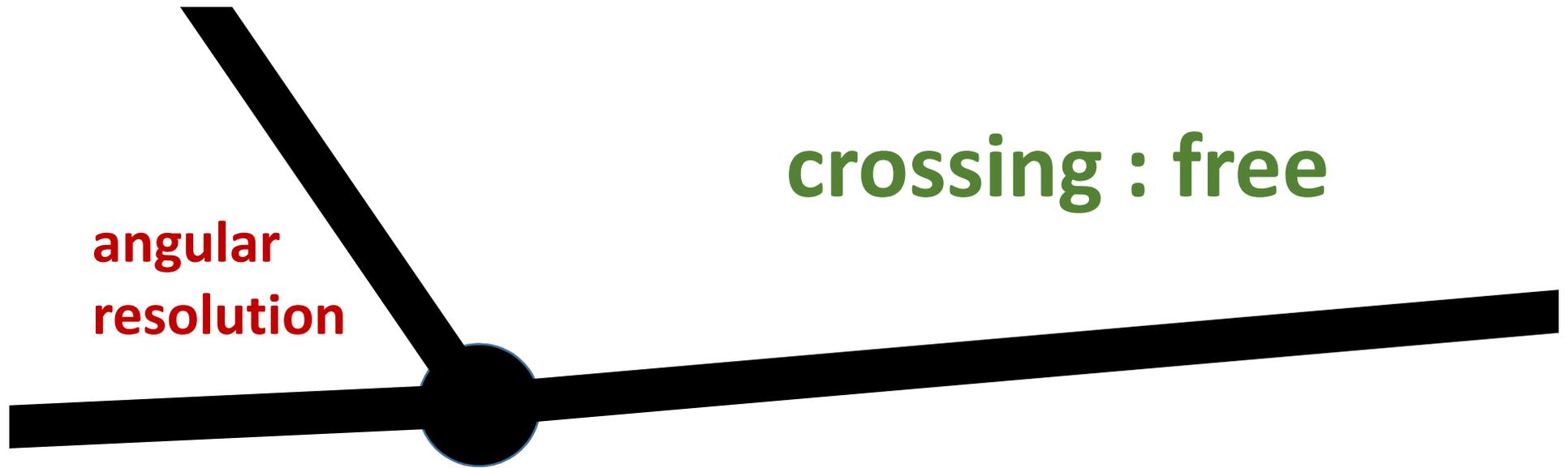


QR ≥ 1.5



Formann et al. '93: Every planar graph can be drawn with **angular resolution** $\Omega(1/d)$; the drawing may be non-planar

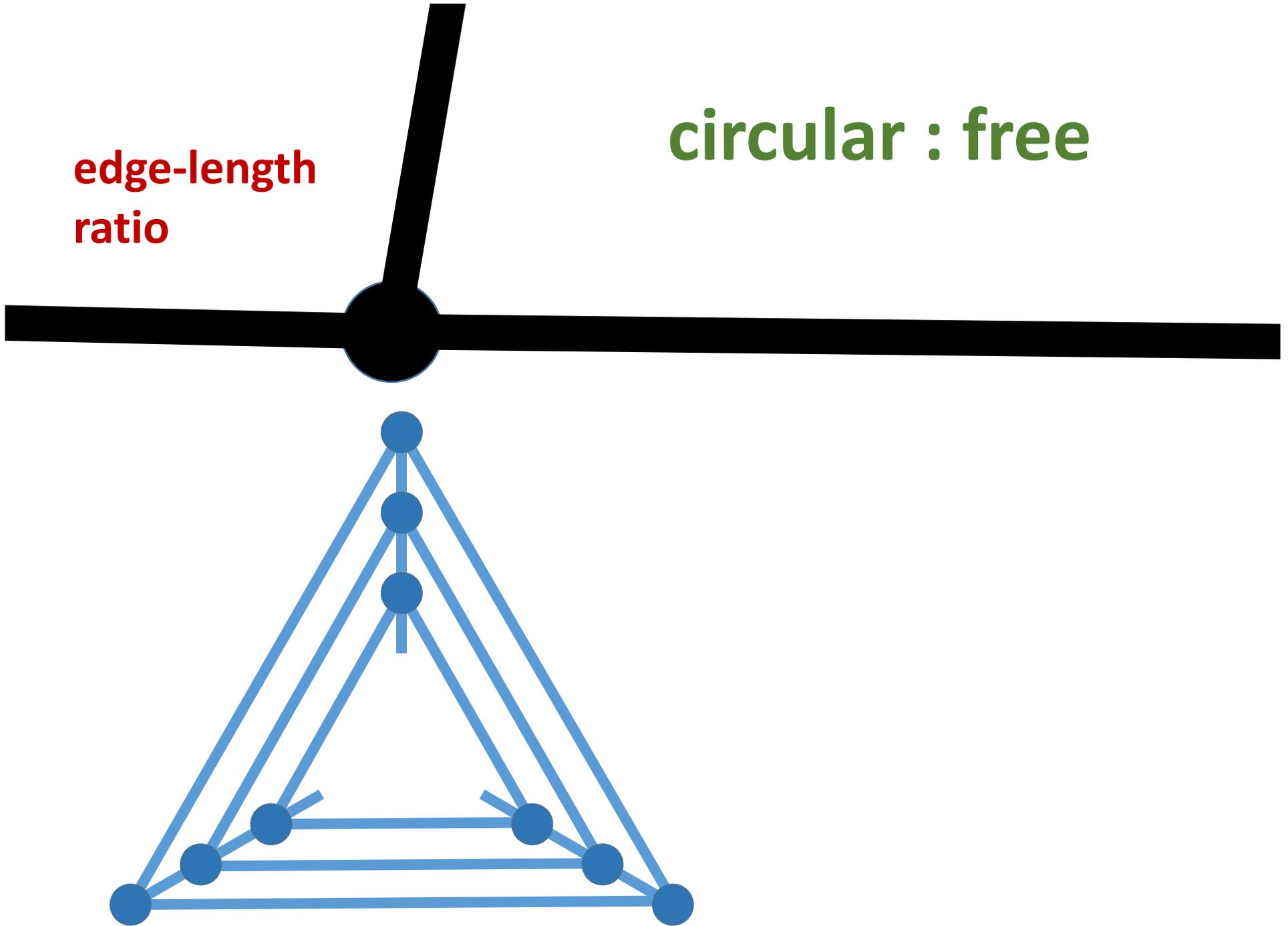
Garg & Tamassia '94: There exists a family of planar graphs with max degree d for which any plane straight-line drawing has **angular resolution** $O(\sqrt{(\log d) / d^3})$



- The quality ratio of crossing versus free drawings for **angular resolution** grows with at least $\sim (1/d) / \sqrt{((\log d) / d^3)}$, so it goes to ∞ as d goes to ∞

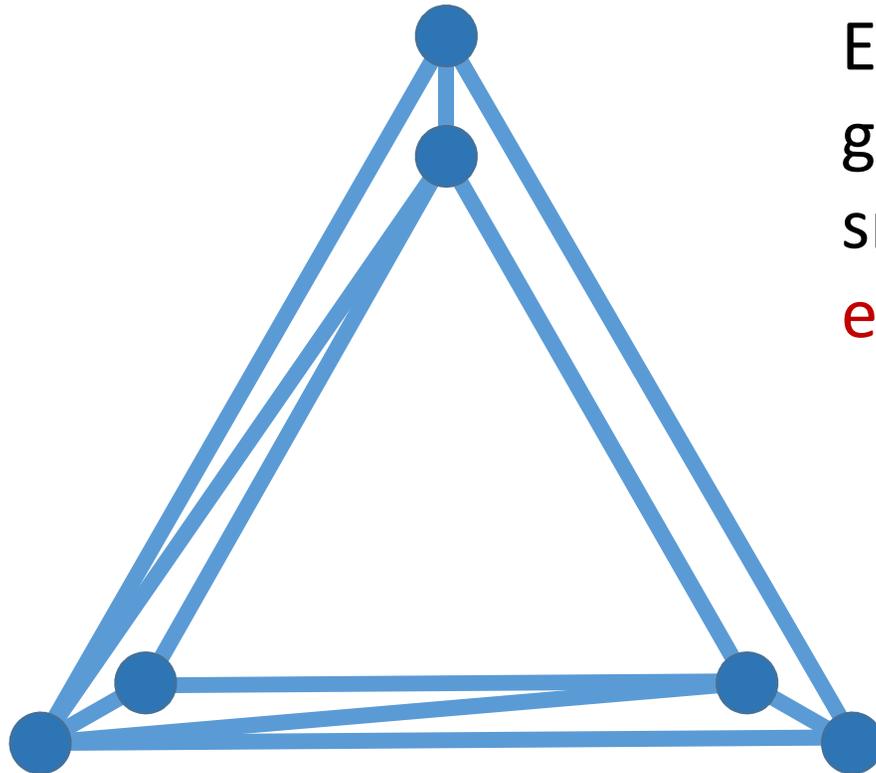
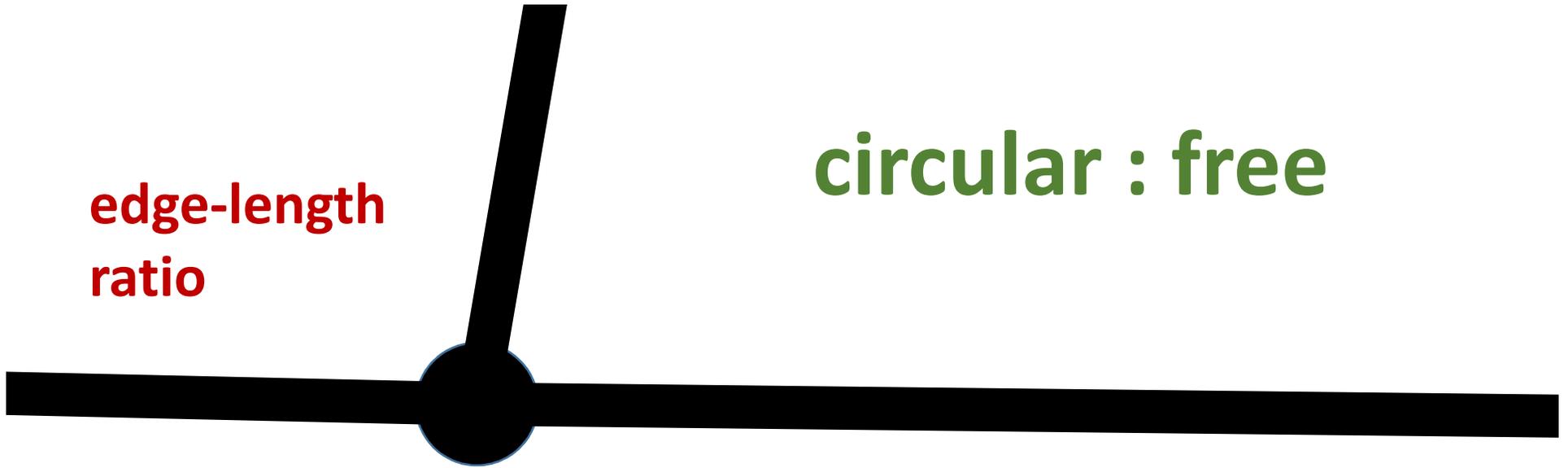
edge-length
ratio

circular : free



edge-length
ratio

circular : free

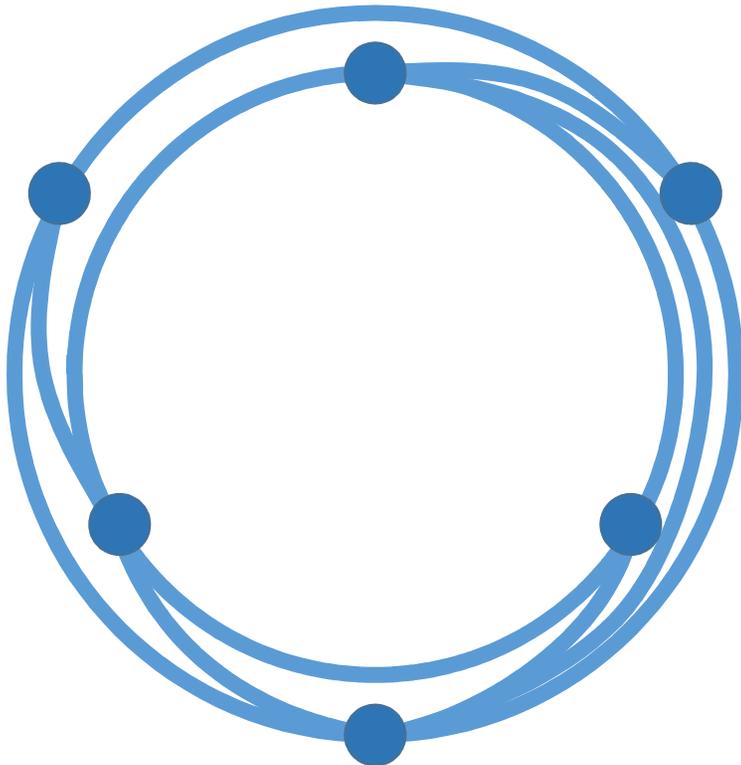


Either the nested triangles
get significantly smaller and
smaller, giving unbounded
edge-length ratio, ...

or the edges between the
triangles must be short,
also giving an unbounded
edge-length ratio

edge-length
ratio

circular : free

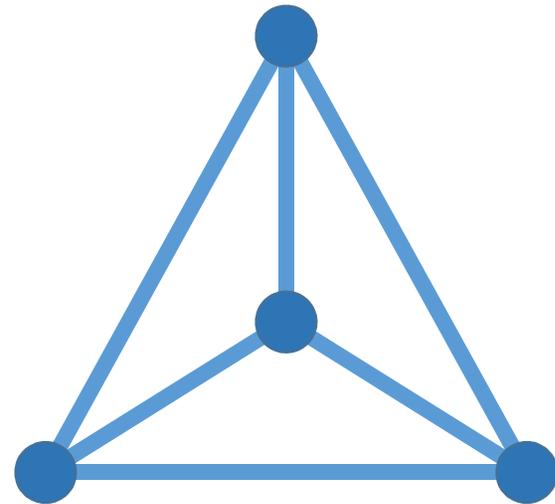
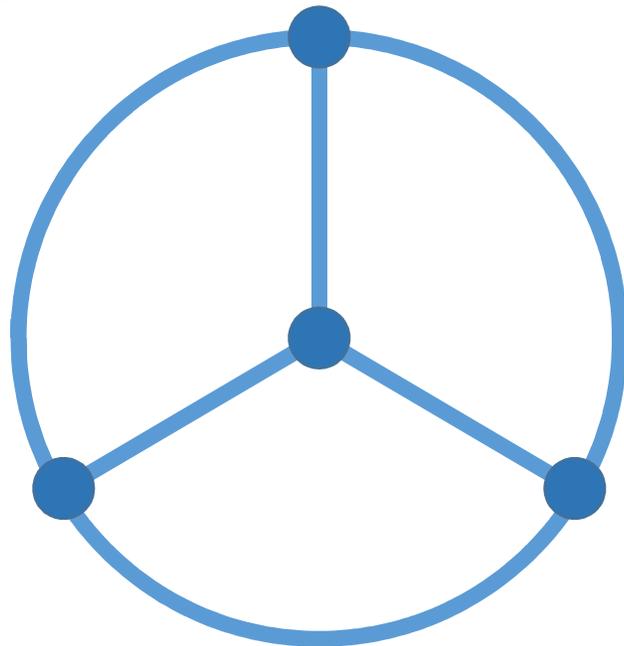


Nested circles can have radii that are arbitrarily close, and the **edge-length ratio** remains ~ 3

→ The quality ratio is unbounded

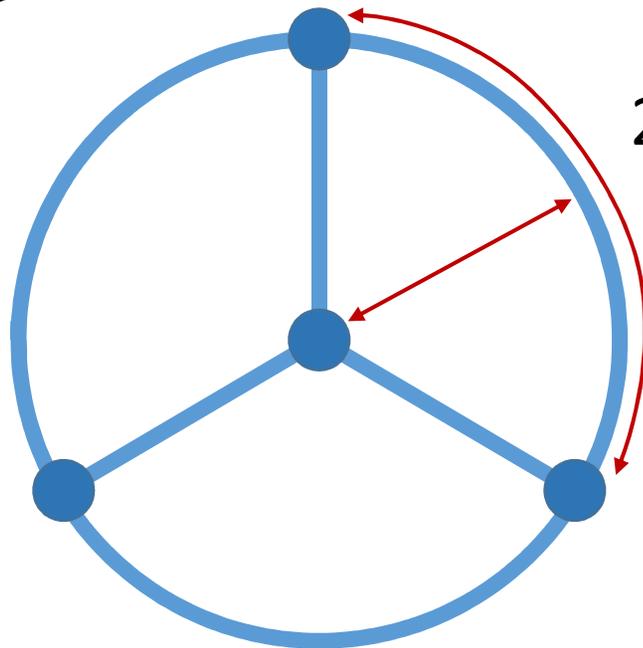
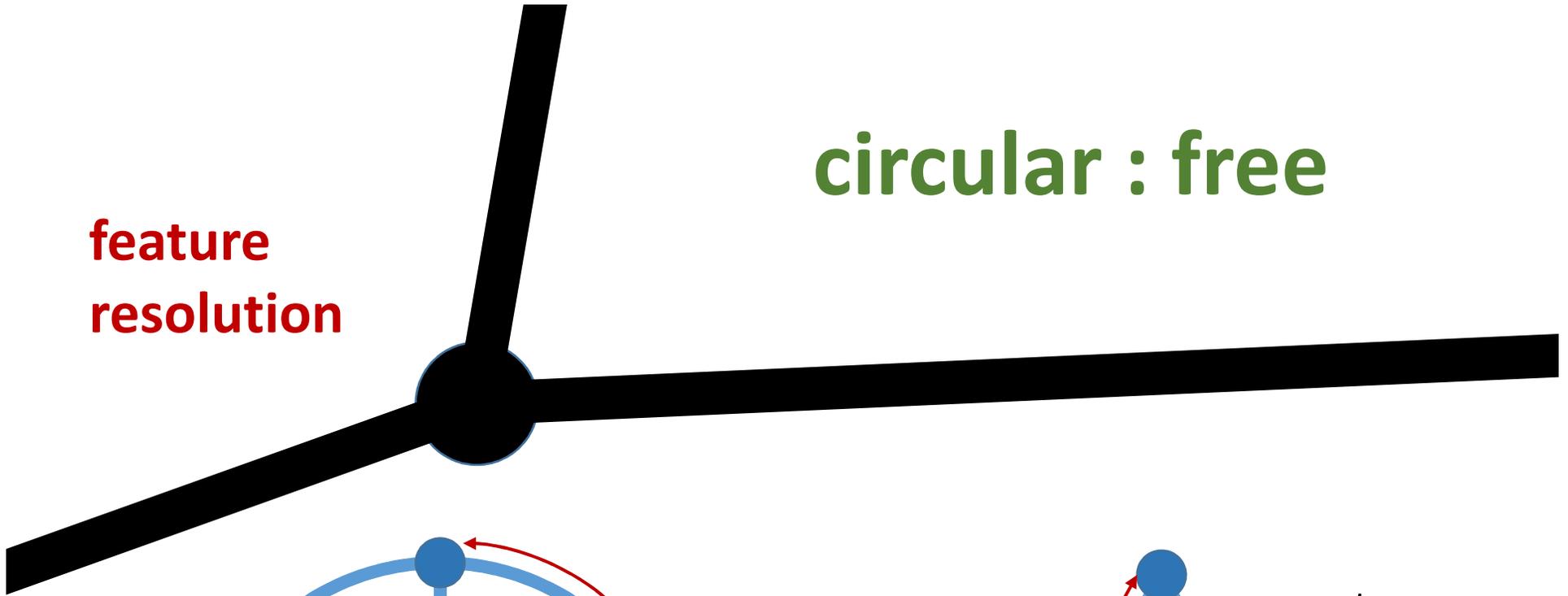
**feature
resolution**

circular : free

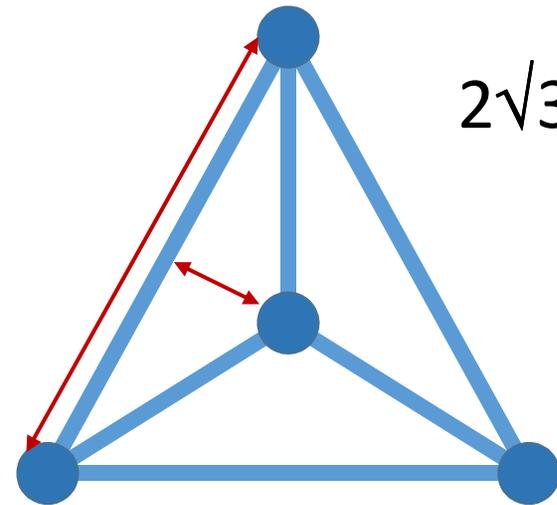


circular : free

feature
resolution



$$2\pi / 3$$



$$2\sqrt{3}$$

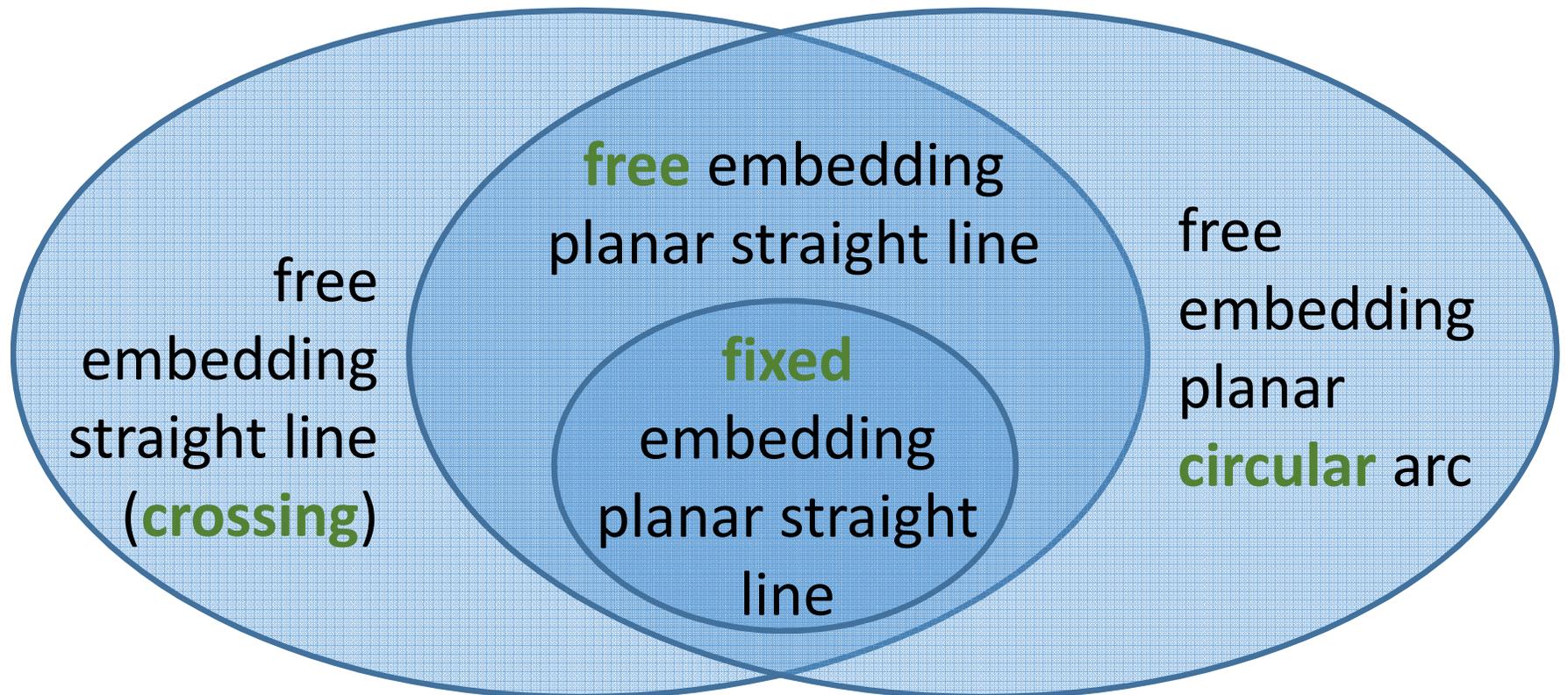
$$QR \geq 3\sqrt{3} / \pi$$

Results

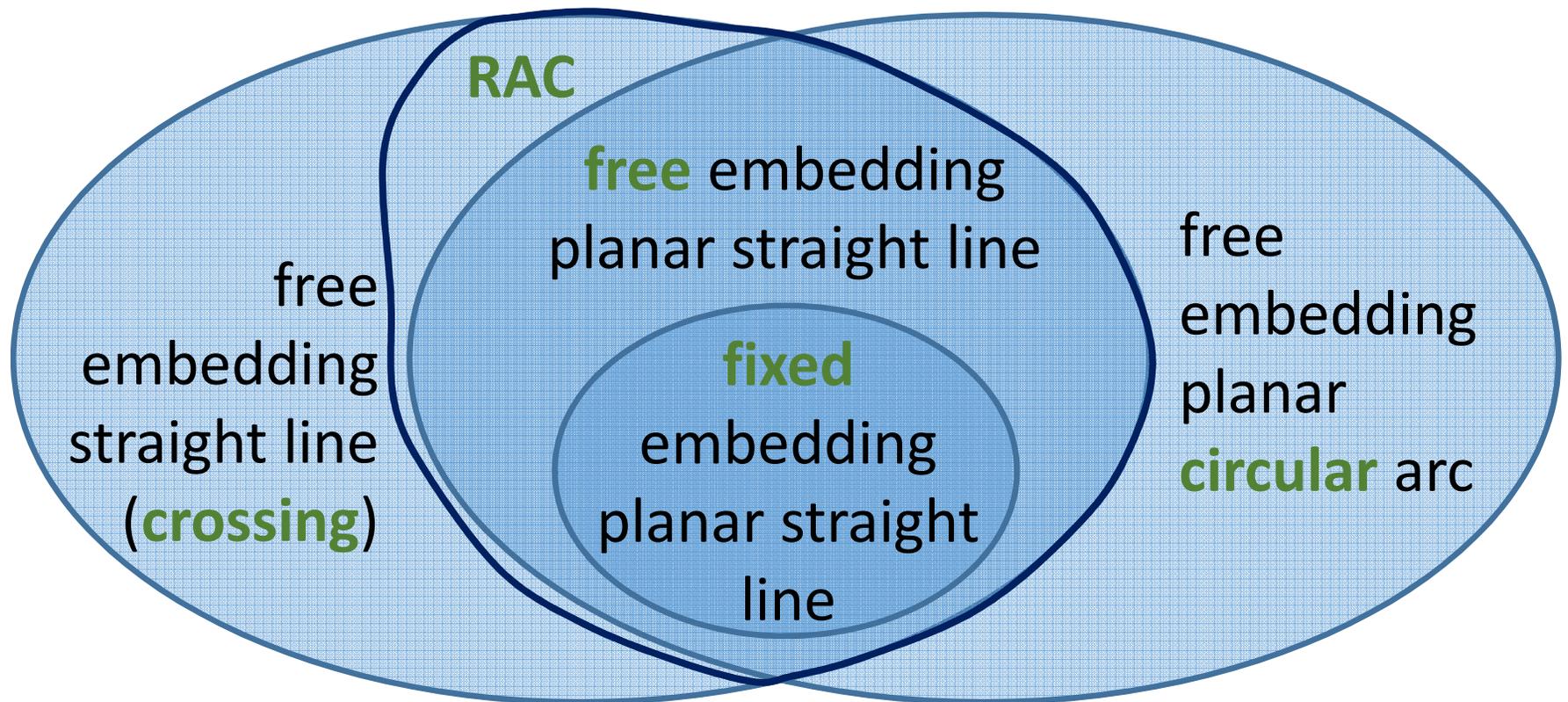
planar graphs

	free : fixed	circ. : free	crossing : free
angular resolution	≥ 12	≥ 4.8	∞
area requirement	∞	∞	∞
edge-length ratio	∞	∞	∞
feature resolution	∞	$\geq 3\sqrt{3} / \pi$	≥ 2.509

Drawing Styles



Drawing Styles



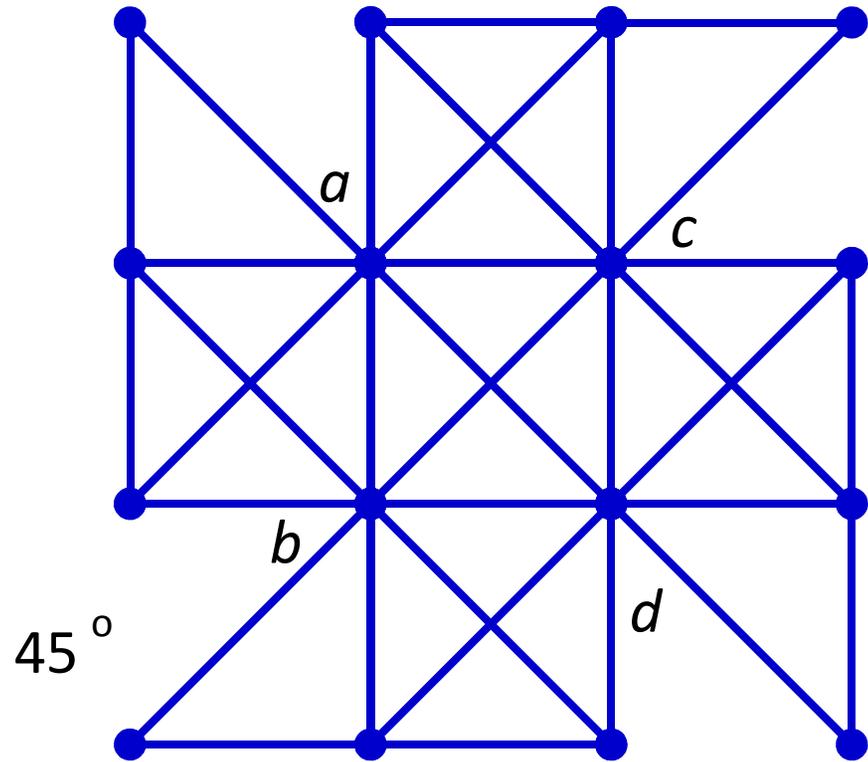
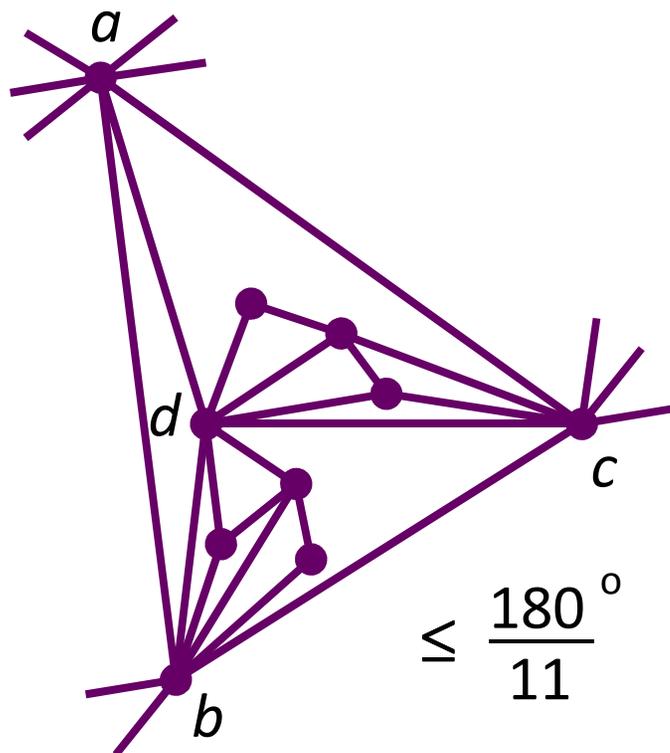
Results

planar graphs

	free : fixed	circ. : free	RAC : free
angular resolution	≥ 12	≥ 4.8	≥ 2.75
area requirement	∞	∞	∞
edge-length ratio	∞	∞	∞
feature resolution	∞	$\geq 3\sqrt{3} / \pi$	≥ 2.509

Results

planar graphs



The quality ratio is at least 2.75



Related results

Subclasses of graphs or drawings

Other quality measures

Other drawing styles

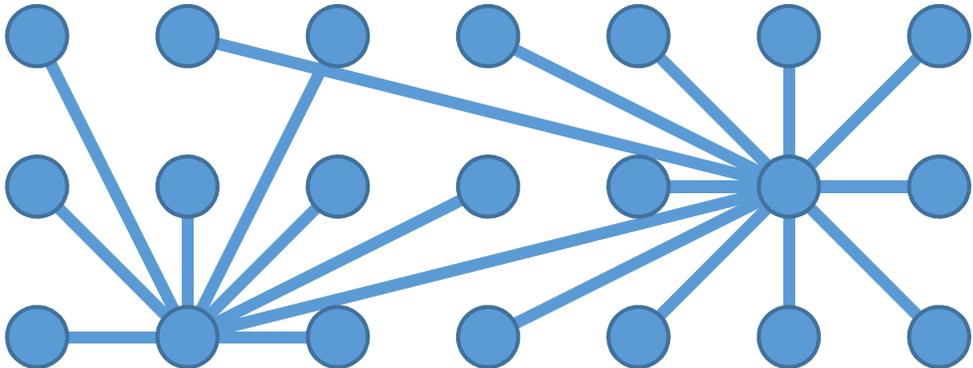
Results

trees

	free : fixed	circ. : free	crossing : free
angular resolution	1	1	1
area requirement	$\geq 16/15$	≥ 1.5	$\geq 22/21$
edge-length ratio	1	1	1
feature resolution	$\geq 1+\epsilon$??	??

Results

trees

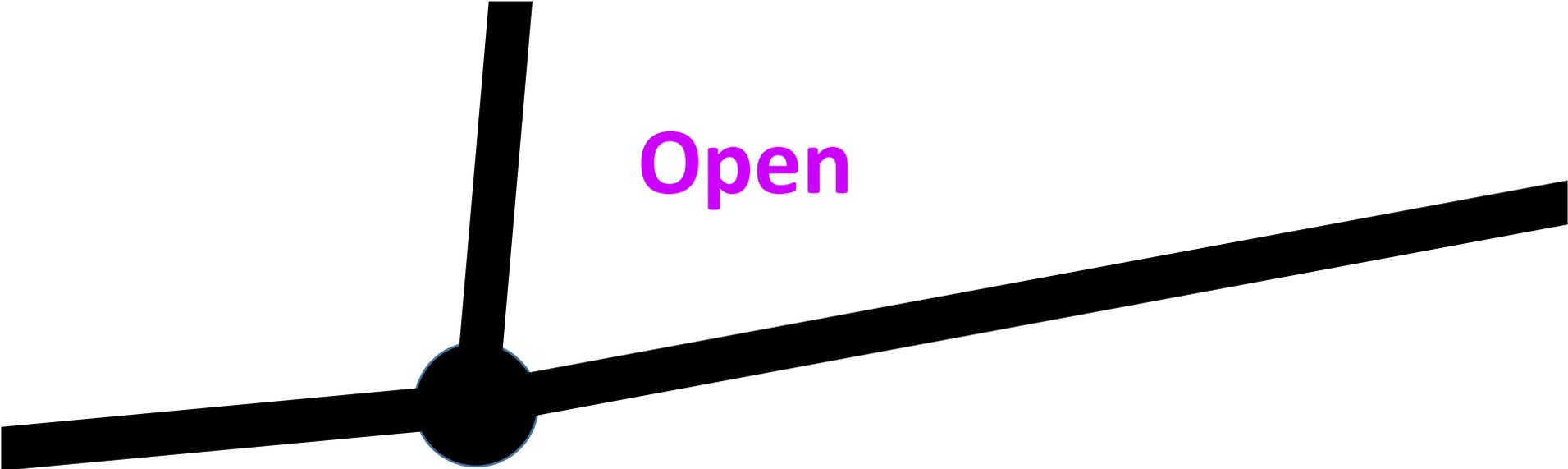


A (9,12)-star cannot be drawn crossing-free on a 7x3 grid

Results / open

unit
edge-length
trees

	free : fixed	circ. : free	crossing : free
angular resolution	??	??	??
area requirement	??	??	??
edge-length ratio	1	1	1
feature resolution	??	??	??

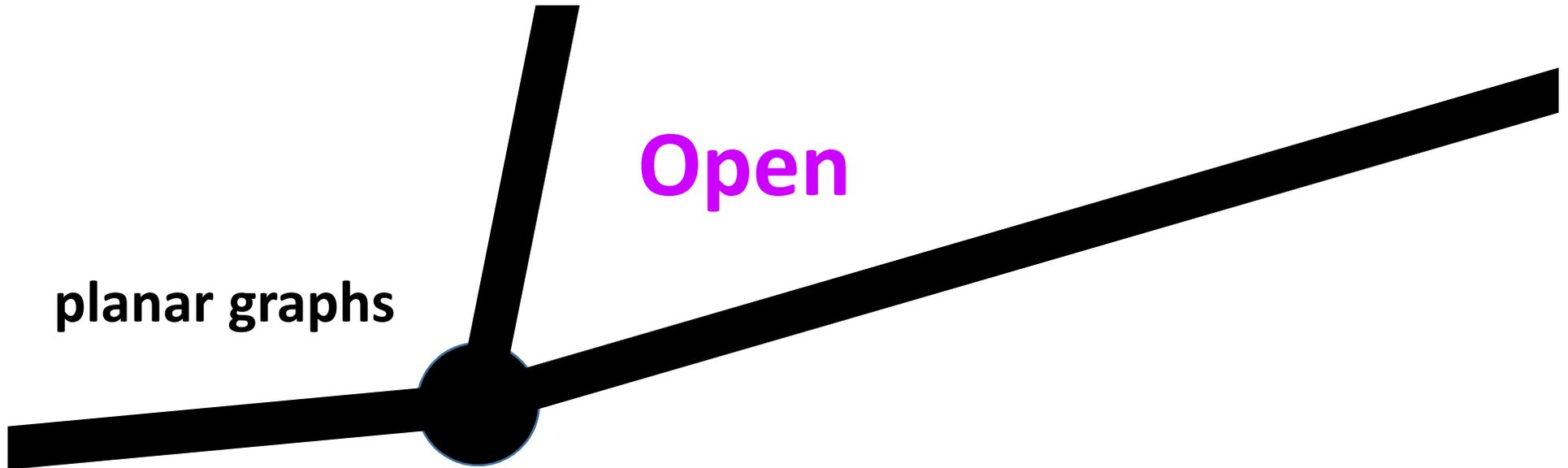


Open

Quality ratios for **straight-line planar** and **one-bend planar** drawings

Quality ratios for **one-bend planar** and **two-bend planar** drawings

Quality ratios for **circular arc planar** and **Bezier curve planar** drawings



Open

planar graphs

1-bend : free 2-bend : 1-bend Bezier : circ

angular resolution

??

??

??

area requirement

??

??

??

edge-length ratio

??

??

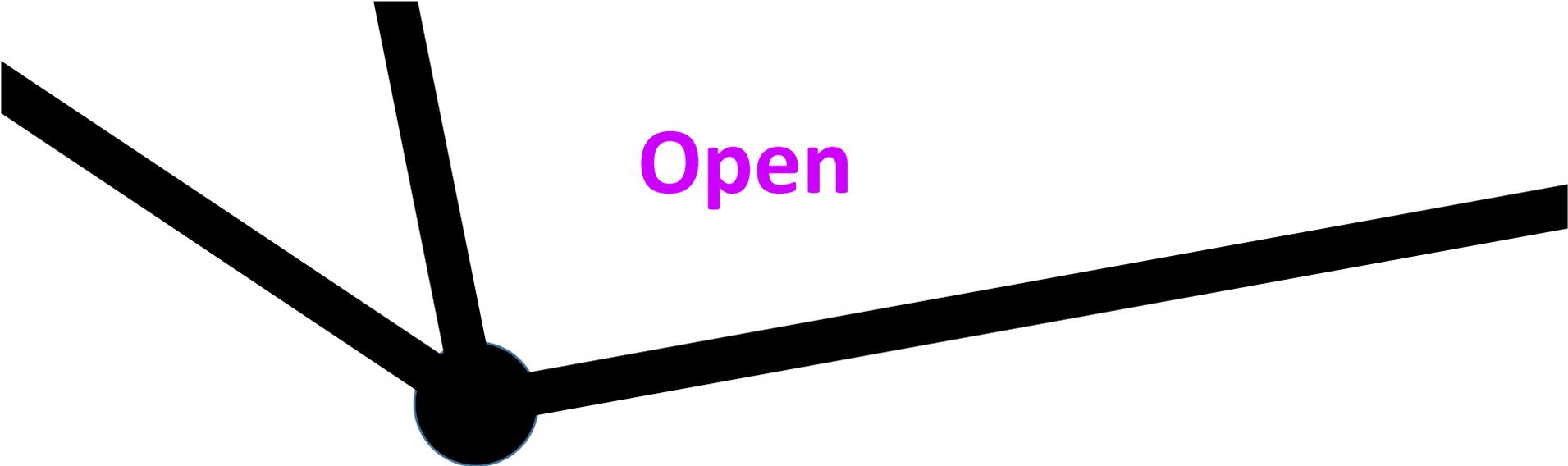
??

feature resolution

??

??

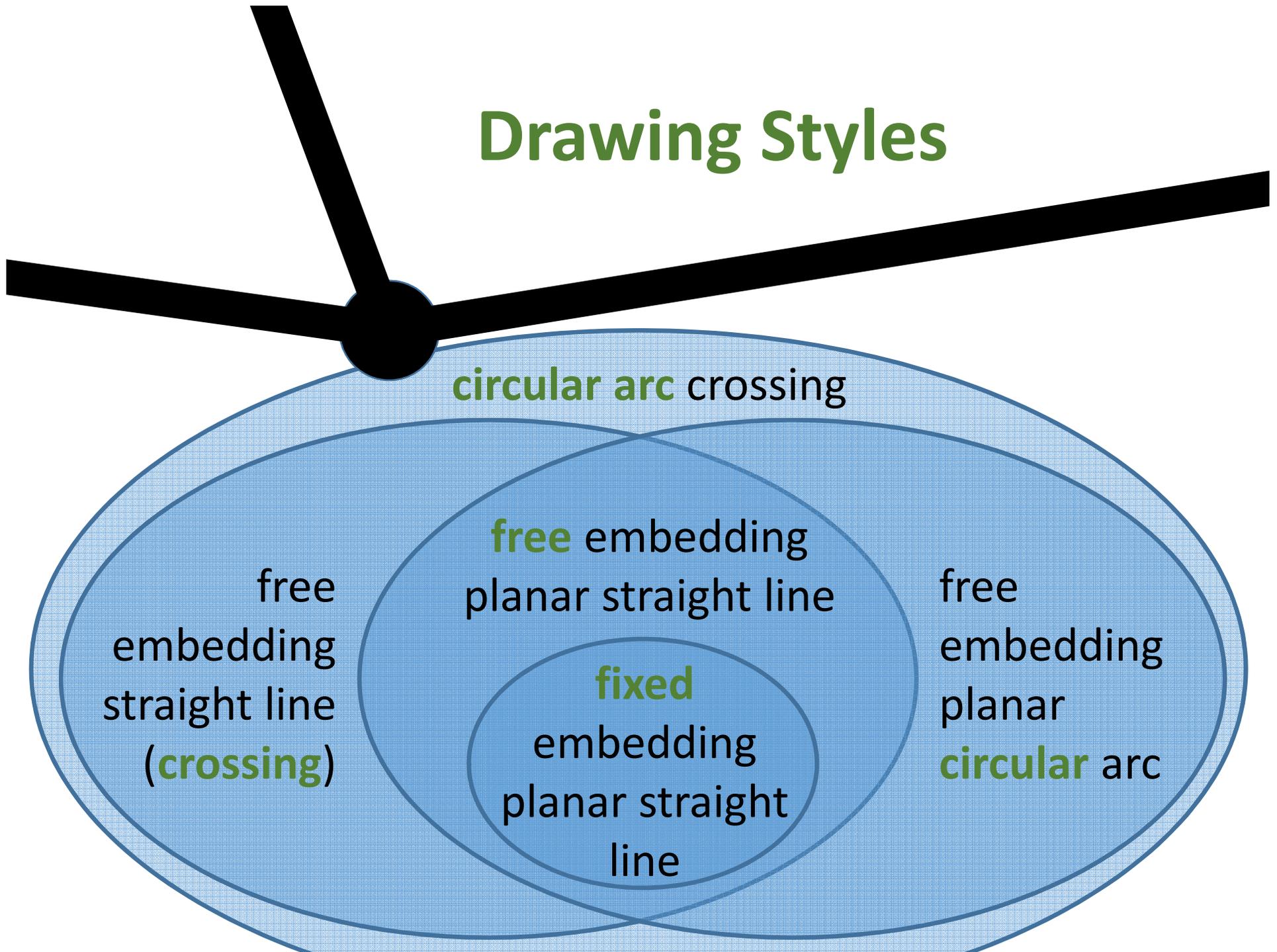
??



Open

Quality ratios for **straight-line crossing** and **circular arc crossing** drawings for general graphs

Drawing Styles



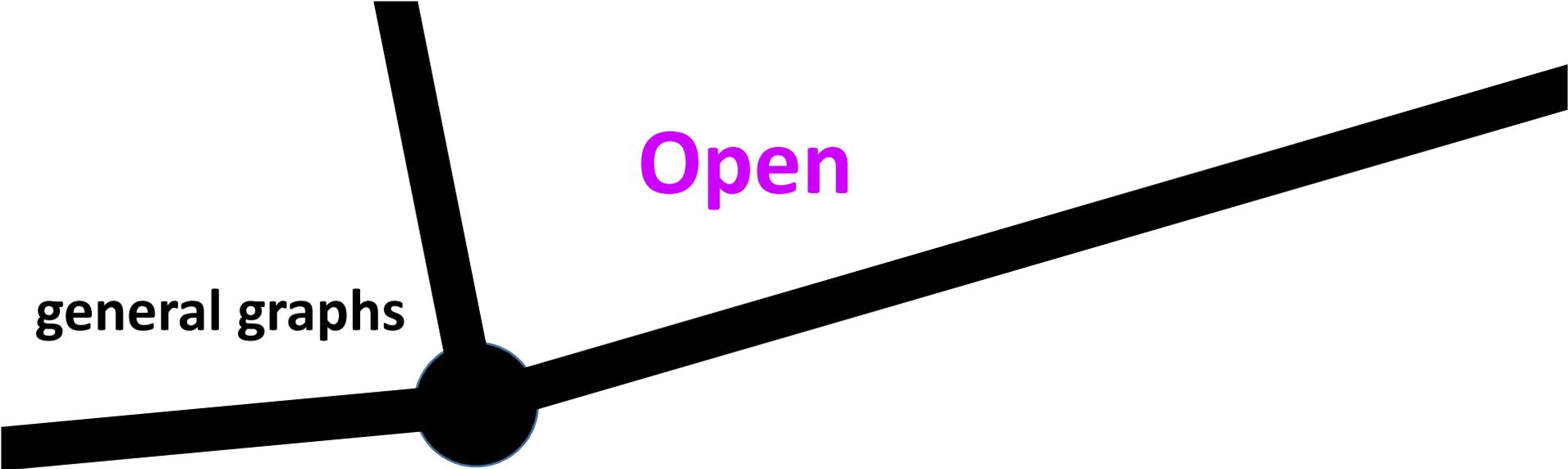
circular arc crossing

free
embedding
straight line
(**crossing**)

free embedding
planar straight line

free
embedding
planar
circular arc

fixed
embedding
planar straight
line



Open

general graphs

circular-crossing : straight-crossing

angular resolution

?? will be $O(1)$

area requirement

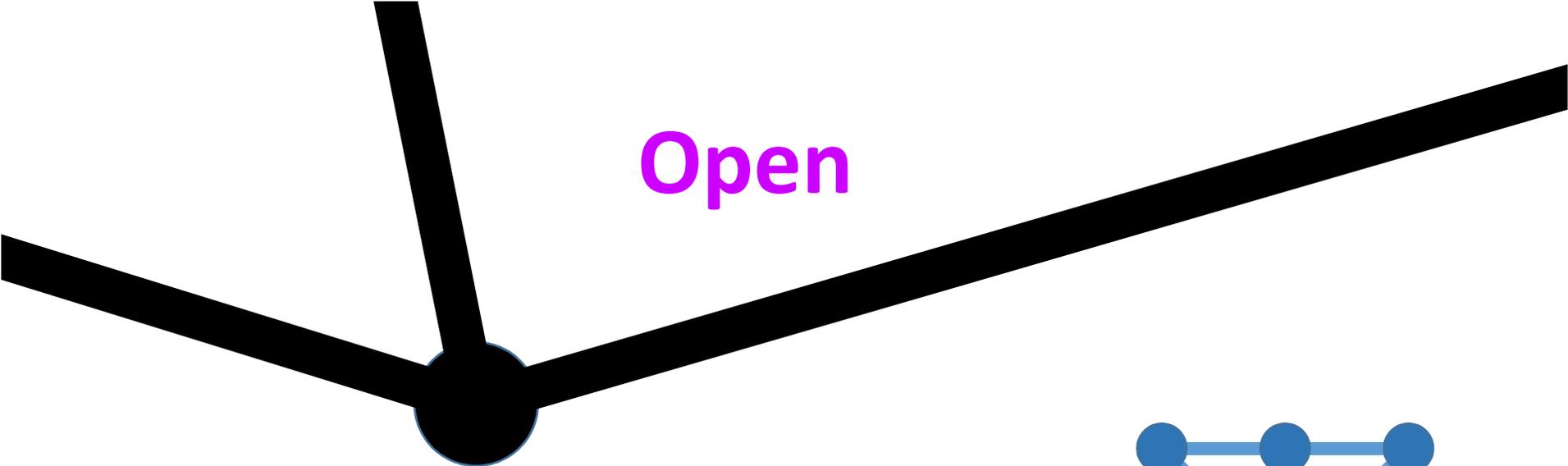
??

edge-length ratio

??

feature resolution

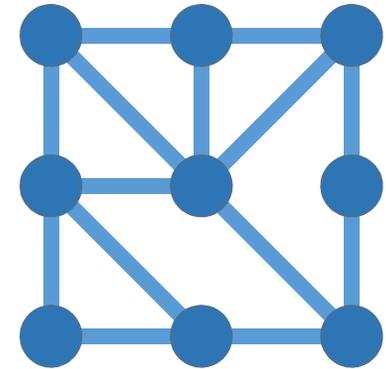
??



Open

Quality ratios for the measure
bounded-face area ratio for all
drawing style pairs

Quality ratios for the measure
bounded-face diameter ratio
for all drawing style pairs



**bounded-face
area ratio 2**
(but can be made 1)

Open

planar graphs

free : fixed

circ. : free

crossing : free

bounded-face
area ratio

??

??

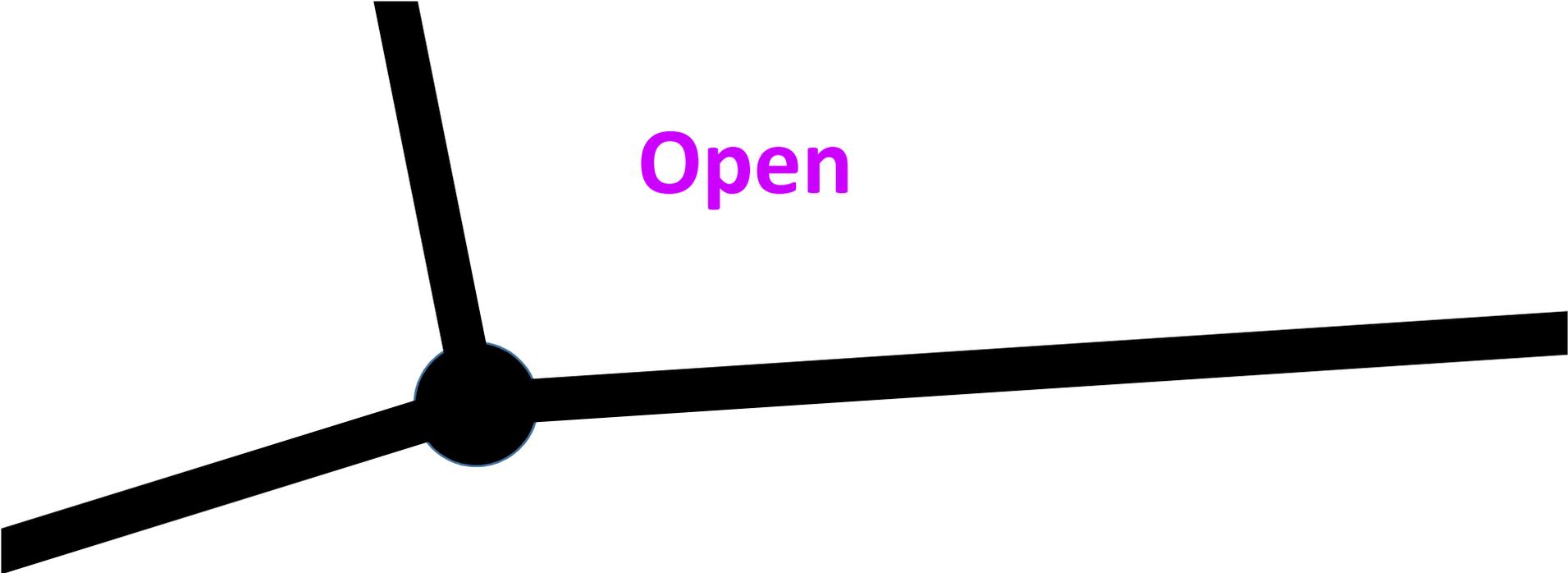
??

bounded-face
diameter ratio

??

??

??



Open

Observe that we have **no constant upper bounds** of quality ratios different from 1

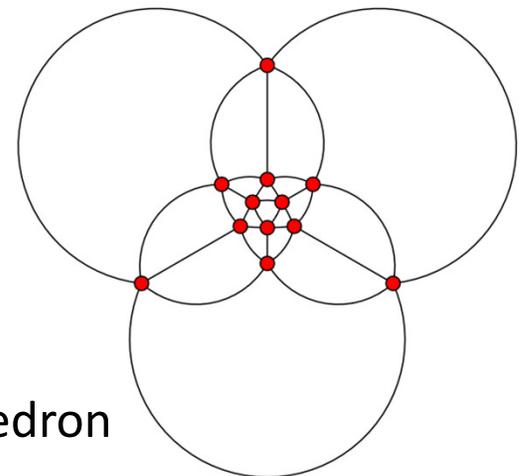
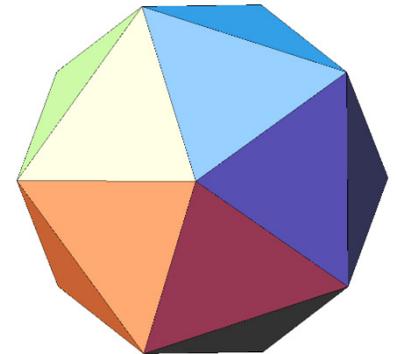
When an entry is ∞ , we are also interested in the **max. asymptotic rate** at which it goes to ∞

The best(??) open problems

No. 1

Quality ratio of circular arc plane drawings and free straight-line plane drawings for angular resolution of planar graphs

- upper bound?
- better lower bound than 4.8 ?



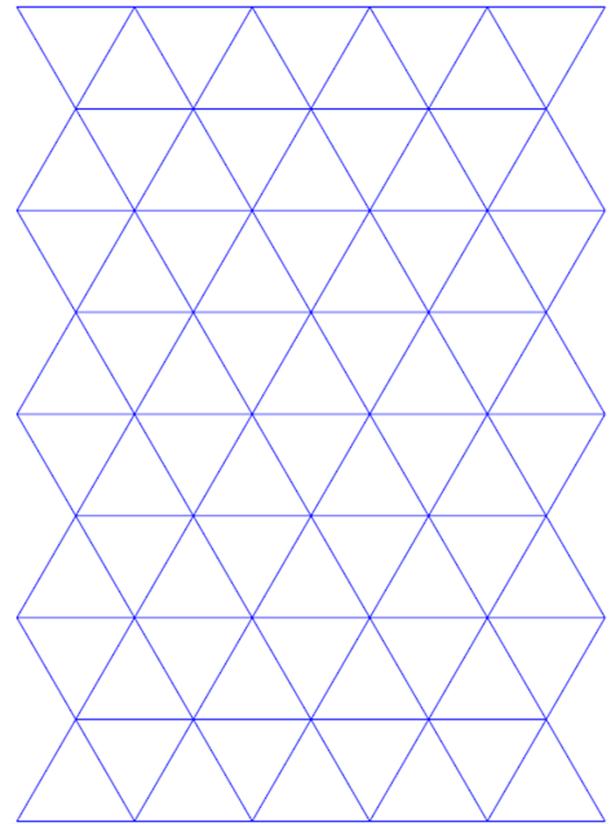
Lombardi drawing of an icosahedron

The best(??) open problems

No. 2

Quality ratio of fixed plane drawings and free straight-line plane drawings for **angular resolution** of unit-edge length trees

- upper bound ?
- lower bound ?

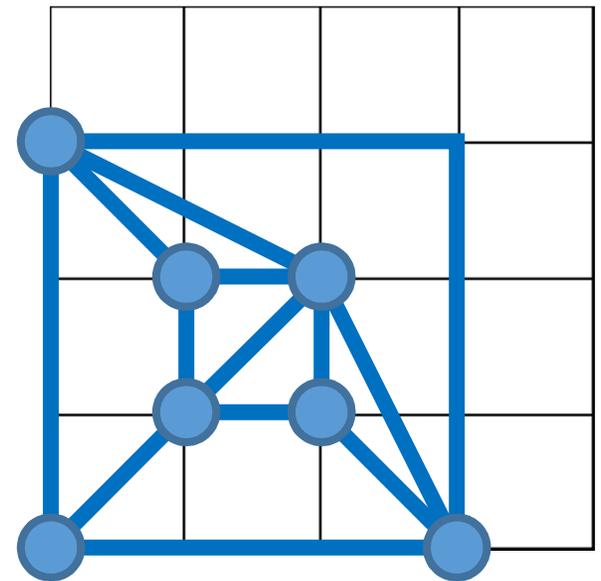


The best(??) open problems

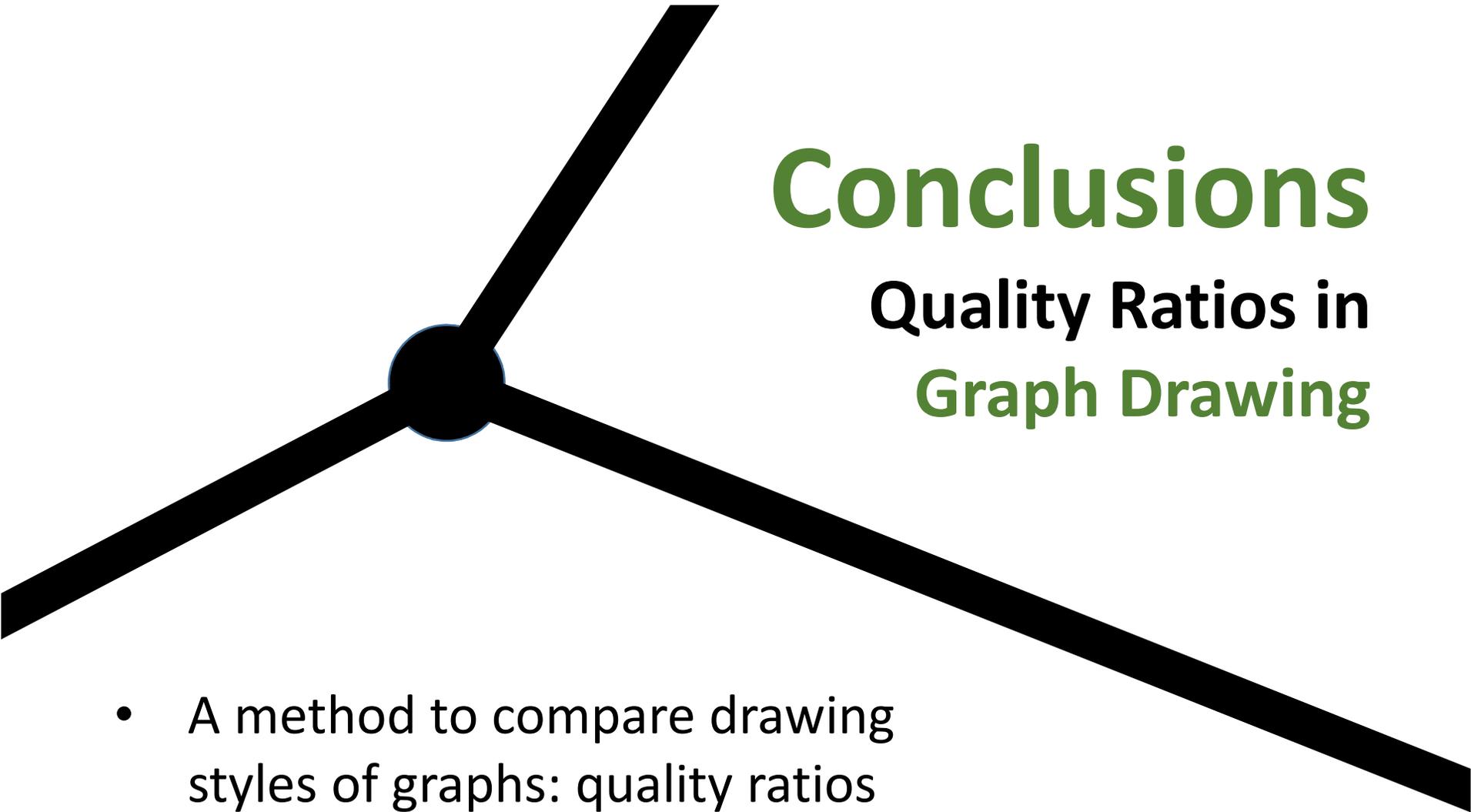
No. 3

Quality ratio of 1-bend plane drawings and free straight-line plane drawings for **grid size** of planar graphs

- upper bound ?
- lower bound ?



Square inside a triangle



Conclusions

Quality Ratios in Graph Drawing

- A method to compare drawing styles of graphs: quality ratios
- Various results for four (six) **quality measures** and four/five (eight) **drawing styles**
- Many **open problems**