Realizing Graphs as Polyhedra

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Recent Trends in Graph Drawing:
Curves, Graphs, and Intersections

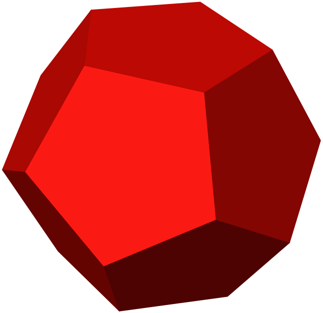
California State University Northridge, September 2015

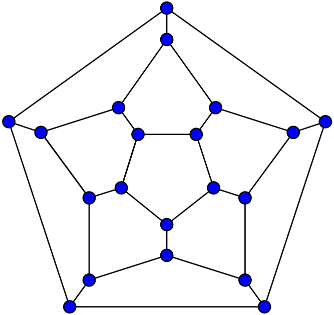
Outline

- Convex polyhedra
  - Steinitz's theorem
  - Incremental realization
  - Lifting from planar drawings
  - Circle packing
- Non-convex polyhedra
- Four-dimensional polytopes

Steinitz's theorem

Graphs of 3d convex polyhedra
= 3-vertex-connected planar graphs
[Steinitz 1922]

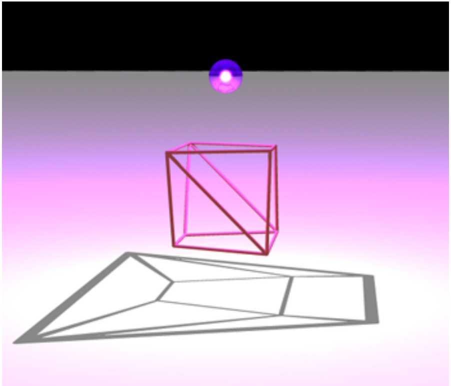




File:Uniform polyhedron-53-t0.svg and File:Graph of 20-fullerene w-nodes.svg from Wikimedia commons

Polyhedral graphs are planar

Projection from near one face gives a planar *Schlegel diagram*



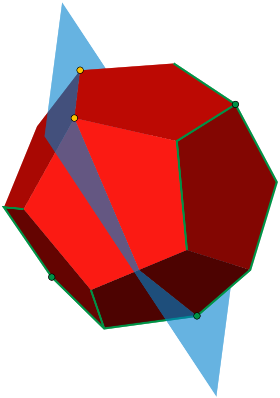
Alternatively, the polyhedral surface itself (minus one point)
is topologically equivalent to a plane

Polyhedral graphs are 3-connected

Theorem [Balinski 1961]: d-dimensional polytopes are d-connected

Proof idea:

- Consider any set C of fewer than d vertices
- Add one more vertex v
- Find linear function f, zero on $C \cup \{v\}$, nonzero elsewhere
- Simplex method finds C-avoiding paths from v to both $\min f$ and $\max f$, and from all other vertices to min or max
- Therefore, C does not disconnect the graph



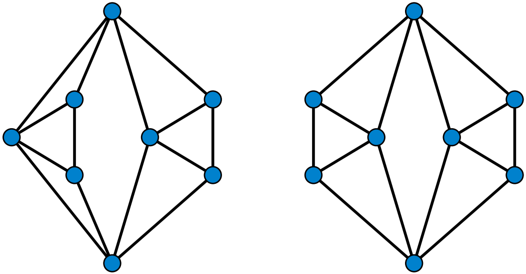
Combinatorial uniqueness of realization

Face cycles do not separate rest of graph:

Any two points not on given face can be connected by a path disjoint from the face (same argument as Balinski)

Non-face cycles do separate one side of cycle from the other (Jordan curve theorem)

So faces are determined by graph structure

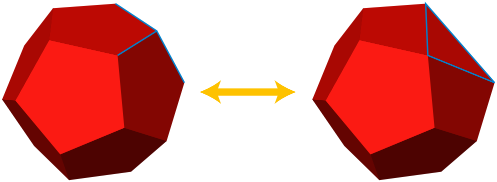


Planar graphs that are not 3-connected can have multiple embeddings

Polyhedral realization of 3-connected planar graphs

Three different methods for proving that all 3-connected planar graphs can be realized as polyhedra:

- Induction
  - Edge deletion or contraction [Barnette and Grünbaum 1969]
  - Δ–Y transformations [Grünbaum 2003; Ziegler 1995]



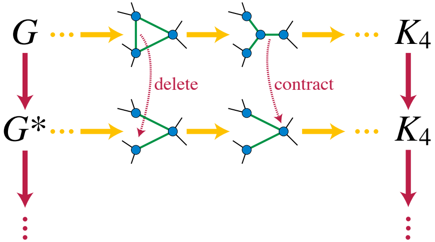
- Find planar projection first, then lift to 3d
- Circle packing

Induction by Δ–Y transformations

All 3-connected planar graphs can be transformed into K_4 by replacing non-separating triangles $\leftrightarrow$ degree-three vertices

[Epifanov 1966; Truemper 1989; Ziegler 1995]

Part 1: If G is ΔY-reducible, so is any minor of G



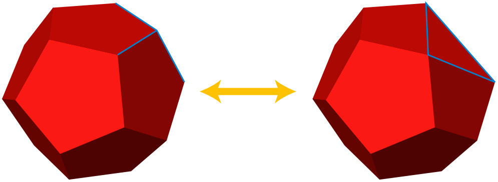
Part 2: Every planar graph is a minor of a grid graph

Part 3: Every grid graph is ΔY-reducible

Realizability from ΔY-reducibility

Find a sequence of Δ–Y steps leading from G to K_4

Reverse the sequence, performing each reversed step geometrically



Problem: leads to low-quality realizations

How do you measure the quality of a polyhedron?

One possibility:

- Require integer coordinates
- Higher quality $\Leftrightarrow$ smaller numbers

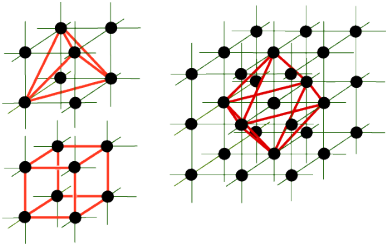


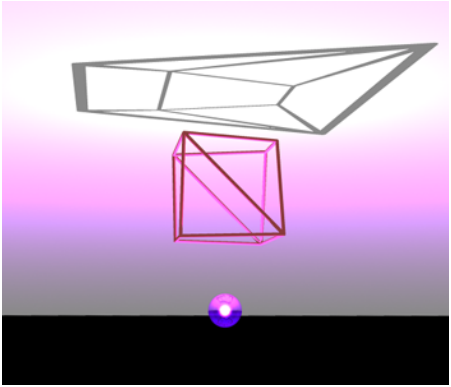
Image by Paul R. Scott from http://paulscottinfo.ipage.com/lattice-points/5regular.html

Repeated multiplication $\Rightarrow$ double exponential coordinates

For simplicial polyhedra, can do many independent induction steps in parallel $\Rightarrow$ exp(polynomial) coords [Das and Goodrich 1997]

Lifting approach for nicer realizations

Reverse projection: turn Schlegel diagram back into a polyhedron

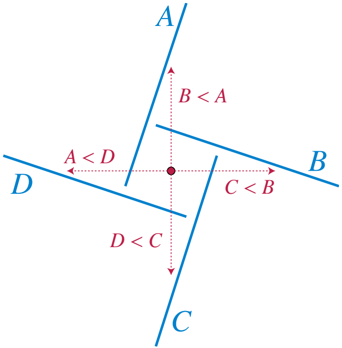


(But it will be easier to reverse perpendicular projection rather than perspective projection)

Necessary conditions for projections of polyhedra

Must subdivide a convex polygon into smaller convex polygons

Visibility ordering from any point must be acyclic [Edelsbrunner 1990]



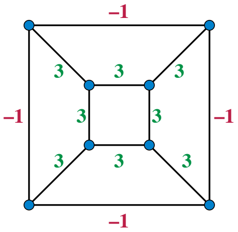
Maxwell–Cremona correspondence

Necessary and sufficient for projection of any polyhedral surface

[Maxwell 1864; Whiteley 1982]

Stress: nonzero edge weights (possibly negative)

Equilibrium stress: at every vertex, weighted sum of neighbor vectors = zero vector



Equilibrium $\Leftrightarrow$ projected surface

Stress $\times$ length $\Rightarrow$ slope difference in perpendicular cross-section

$> 0 \Leftrightarrow$ convex

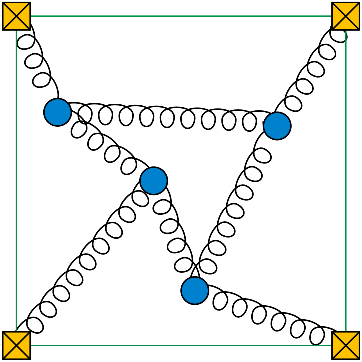
$< 0 \Leftrightarrow$ concave

Tutte's spring embedding

Given a 3-vertex-connected planar graph:

- Fix the outer face vertices in convex position
- Replace edges by ideal springs (Hooke's law: force $=$ length)
- Equilibrium state: planar, all faces convex

[Tutte 1963]



Tutte + Maxwell–Cremona

To find Tutte embedding, solve linear equations

interior vertex = average of neighbors

So these vertices have equal-weight equilibrium stress

Outer vertices might not be in equilibrium
(unless it's a triangle)

If there is a triangular face, use it as the outer face
and lift using Maxwell–Cremona

Else the dual graph has a triangle, realize it then dualize

[Onn and Sturmfels 1994; Eades and Garvan 1996]

How small can the coordinates be?

Tutte + Maxwell–Cremona method can be

- at least exponential [Eades and Garvan 1996]
- at most n^{169n^3} [Onn and Sturmfels 1994]
- at most 43^n when there is a triangular face and at most 2^{18n^2} otherwise [Richter-Gebert 1996]

Incremental convex drawing + Maxwell–Cremona $\Rightarrow$ $O(n \log n)$-bit rational-number coordinates [Chrobak et al. 1996]

Extend Tutte + MC to quadrilateral or pentagon outer faces (instead of using duality) $\Rightarrow$ $O(188^n)$ [Ribó Mor et al. 2011]

Tighten analysis by bounding max # spanning trees in planar graphs $\Rightarrow$ $O(148^n)$ [Buchin and Schulz 2010]

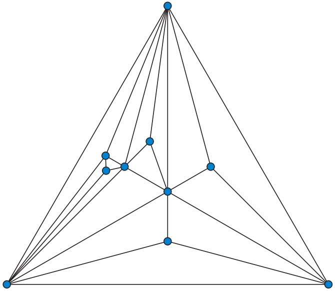
Surveyed by Rote [2012]

Open: Can we do better than single-exponential?

A special case with polynomial coordinates

Stacked polyhedron: formed by gluing tetrahedra face-to-face

Combinatorial construction: repeatedly subdivide triangular face into three triangles

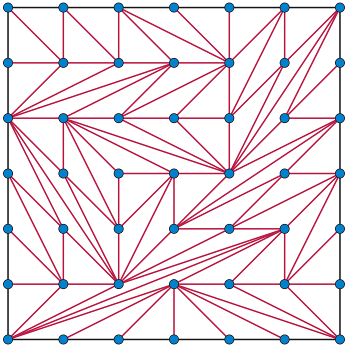


Use incremental construction (with careful subdivision ordering) to build a special equilibrium stress, then lift

$\Rightarrow O(n^4) \times O(n^4) \times O(n^{18})$ [Demaine and Schulz 2011]

Another case with subexponential coordinates

Triangulate points in a $\sqrt{n} \times \sqrt{n}$ grid

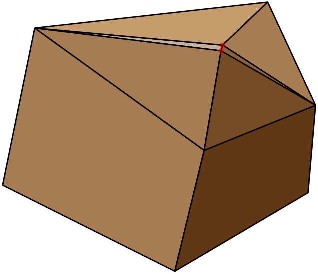


Can lift to a polyhedron with coordinates in
$O(n^3) \times O(n^5) \times \exp O(\sqrt{n} \log n)$ [Pak and Wilson 2013]

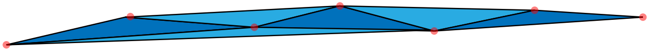
Same method realizes any n-vertex polyhedral graph in
$O(n^3) \times O(n^5) \times \exp O(n \log n)$

Alternative quality: vertex resolution

Exponential coordinates could mean: some vertex distances are much smaller than others



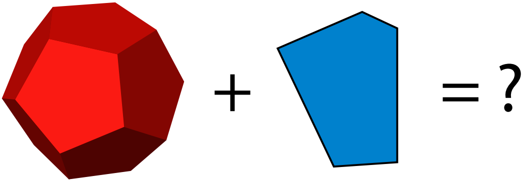
However, every polyhedron can be realized with all vertices ≥ 1 unit apart in an $O(n) \times O(1) \times O(1)$ bounding box [Schulz 2011]



Choosing the shape of a face

Tutte lets us find a convex polygonal subdivision in which the outside face shape can be chosen arbitrarily

But it doesn't necessarily lift to a polyhedron



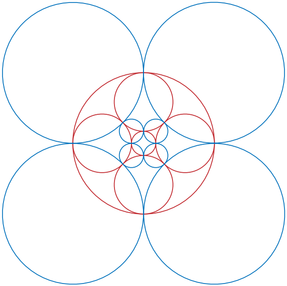
Every polyhedral graph can be realized with a specified shape for one of its faces (proof idea: induction) [Barnette and Grünbaum 1970]

Circle packing theorem (primal-dual form)

For any 3-connected planar graph:

- Represent vertices and faces as circles (in the plane, or on a sphere)
- Adjacent graph vertices $\Leftrightarrow$ tangent circles
- Adjacent graph faces $\Leftrightarrow$ tangent circles
- Adjacent vertex-face pairs $\Leftrightarrow$ perpendicular circles

[Koebe 1936; Andreev 1970; Thurston 2002; Brightwell and Scheinerman 1993; Stephenson 2005]



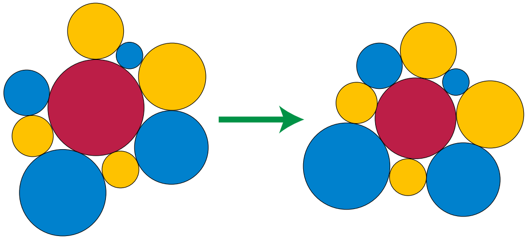
Blue circles represent the vertices of a cube

Red circles represent the faces of a cube

Finding a circle packing

Numerical iteration:

- Maintain circle radii but not positions
- Repeatedly adjust each circle's radius so other circles wrap around it exactly once



- Converges quickly to unique solution

[Collins and Stephenson 2003; Mohar 1993]

Despite being the solution to a system of polynomial equations, closed form formulas do not exist [Bannister et al. 2014]

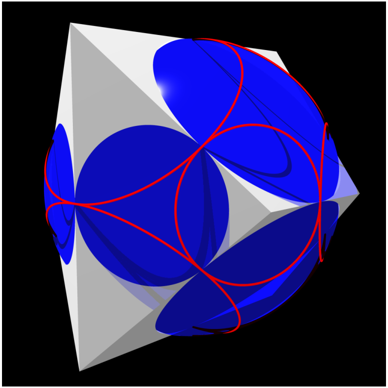
Midsphere realization

Pack circles on a sphere instead of the plane

Slice by a plane through each face-circle

Vertex-circles become horizons of visibility from each vertex

Edges are tangent to sphere



Symmetry display

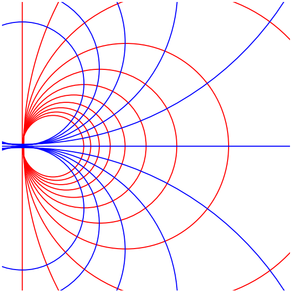
Circle packings are preserved by Möbius transformations

$$q \mapsto \frac{aq + b}{cq + d}$$

Using generalized linear programming, find transformation of sphere to maximize the minimum circle radius

Resulting polyhedron has all the symmetries of the underlying graph

[Bern and Eppstein 2001; Hart 1997]



Möbius transformation of a square grid

Circumspheres and inspheres

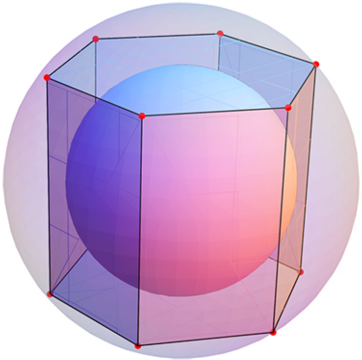


Image from Joe O'Rourke, "Bi-spherical polyhedra", http://mathoverflow.net/questions/140270/bi-spherical-polyhedra

Inscribable = has a circumsphere touching all vertices

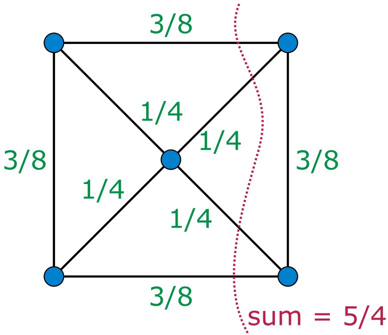
Circumscribable = has an insphere tangent to all faces

Equivalent under planar duality and geometric polarity

Inscribability

Convex realizations with all vertices cospherical $\Leftrightarrow$ edge weights satisfying

- $0 <$ all weights $< 1/2$
- Each vertex has weight sum $= 1$
- Each nontrivial cut has weight sum > 1



Proof idea: weights $\mapsto$ angles between circles in a packing

[Rivin 1996; Dillencourt and Smith 1996]

Outline

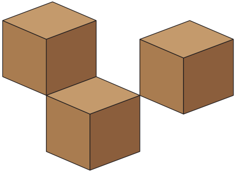
- Convex polyhedra
  - Steinitz's theorem
  - Incremental realization
  - Lifting from planar drawings
  - Circle packing
- Non-convex polyhedra
- Four-dimensional polytopes

Difficult definitional issues

Polyhedron = boundary of a solid, with finitely many flat slides

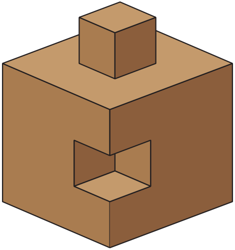
But this still leaves many questions

Are edge-to-edge or point-to-point contacts ok?



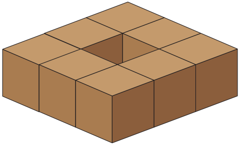
Can the solid be unbounded?

Must the faces be simple polygons?



Can faces meet more than once?

Are flat angles allowed?



Topological sphere or nonzero genus?

The Császár polyhedron

Model by David S. Gunderson at U. Manitoba, from
http://home.cc.umanitoba.ca/∼gunderso/pages/models.html

One of 72 torus realizations of the complete graph K_7

[Császár 1949; Bokowski and Eggert 1991; Lutz 2001]

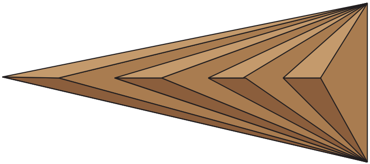
Open: Do any larger complete graphs have polyhedral realizations?
[Ziegler 2008]

Upward star-shaped polyhedra

[Hong and Nagamochi 2011]

Geometrically:

- Topological a sphere
- Faces are star-shaped
- All but one face points up
- No coplanar faces share two vertices

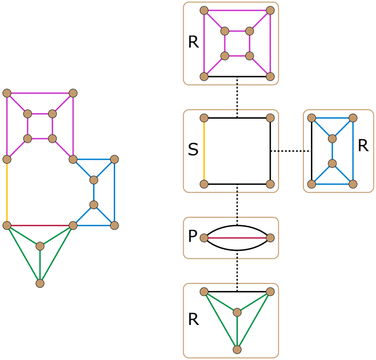


Graph-theoretically:

- 2-connected plane graph
- Min degree ≥ 3
- Two faces that share an edge or three vertices, can't share two outer vertices
- Two pairs of faces that share an edge or three vertices can't both share the same two vertices

Realizing upward star-shaped polyhedra

- Find 3-connected components (nodes of the SPQR tree)
- Induct on # components
- Use Steinitz to realize each component
- Use projective transformations to fit component into rest of polyhedron



Partial results on topological spheres

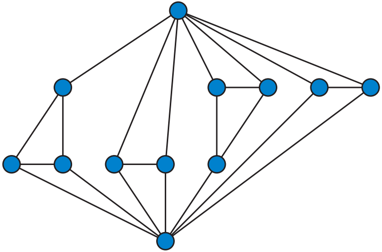
A 3-regular planar graph is realizable if and only if it is 2-vertex-connected and has no P-node in SPQR tree

A plane graph with all face lengths ≤ 4 is realizable if and only if it is 2-connected, has min degree ≥ 3, is simple, and has no "bad triangle"

Both cases can be realized as upward star-shaped polyhedra

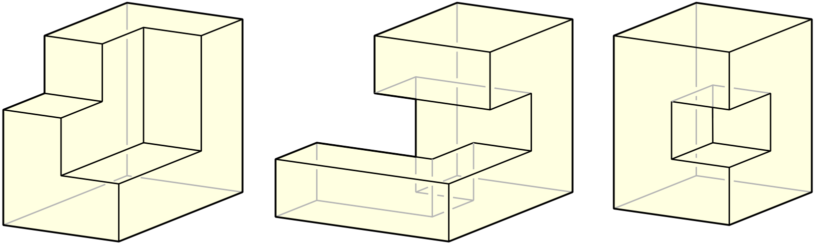
[Hong and Nagamochi 2011]

But...



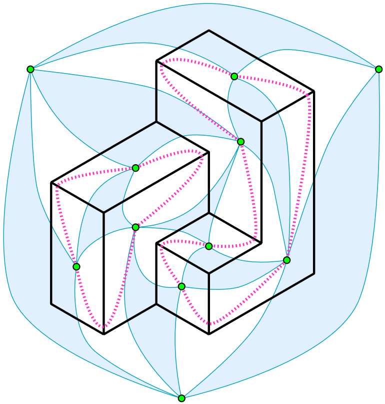
There exist polyhedral spheres whose realizations cannot be upward star-shaped

Simple orthogonal polyhedra



Topological spheres with three orthogonal edges at each vertex $\Leftrightarrow$
3-regular 2-vertex-connected planar bipartite graphs
whose SPQR trees have no P-nodes [Eppstein and Mumford 2014]

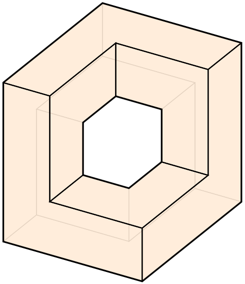
Realizing orthogonal polyhedra



- Split dual triangulation by separating trangles into 4-connected components
- By induction, form cycle cover with one edge per outside-colored triangle
- Cycle structure $\Rightarrow$ which angles are 60° and 120° in axonometric drawing
- Face coords: topological order of DAG derived from dual graph and cycles
- Glue pieces together

Non-spherical orthogonal polyhedra

Not much is known, but...



Manifolds embedded in 3d must be orientable

Graphs drawn in 3d with three perpendicular edges at each vertex form manifolds (with self-crossing planar cycles as faces)

Graph is bipartite $\Leftrightarrow$ manifold is orientable [Eppstein 2013]

$\Rightarrow$ 3-regular orthogonal polyhedra must have bipartite graphs

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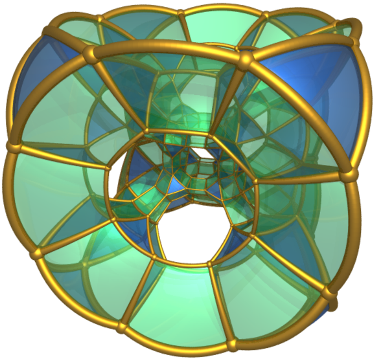
The graphs of 4-polytopes



Include dense graphs e.g. all complete graphs of ≥ 5 vertices (neighborly polytopes)

Face structure is not uniquely determined by graph

Given face structure, NP-hard to test realizability (more precisely $\exists\mathbb{R}$-complete) [Richter-Gebert and Ziegler 1995]

Open: What is the complexity of recognizing the graphs of 4-polytopes?

Simple (4-regular) 4-polytopes

Graph determines unique face structure
[Blind and Mani-Levitska 1987]

Linear function on polytope determines *acyclic unique-sink orientation* of graph, recognizable as orientations minimizing

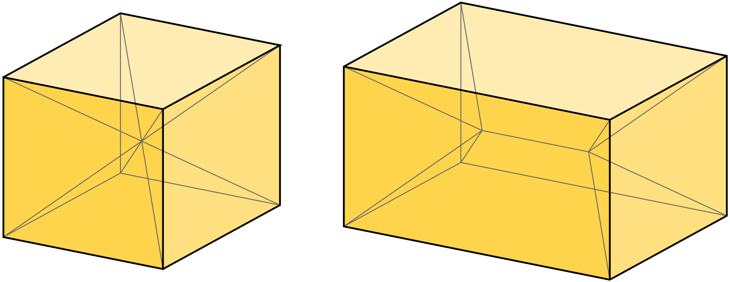
$$\sum_{v} 2^{d_{\mathrm{in}}(v)} = \#\text{ faces of polytope}$$

Faces $\Leftrightarrow$ regular subgraphs whose vertices are minimal for some acyclic unique-sink orientation [Kalai 1988]

Realizability can be tested in polynomial time via linear programming [Friedman 2009]

Treetopes

Polytopes with a special base facet (shown outermost in these Schegel diagrams) such that no 2d face is disjoint from the base



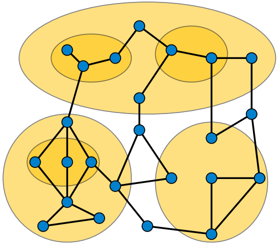
Pyramid over cube (left) and prism over square pyramid (right)

Parts of the graph incident to non-base vertices form a tree

Recognizable in polynomial time using techniques related to clustered planarity [Eppstein 2016]

Clustered planar graph drawing

Input: a graph together with a hierarchical clustering (sets of vertices with each pair either disjoint or subset-superset)



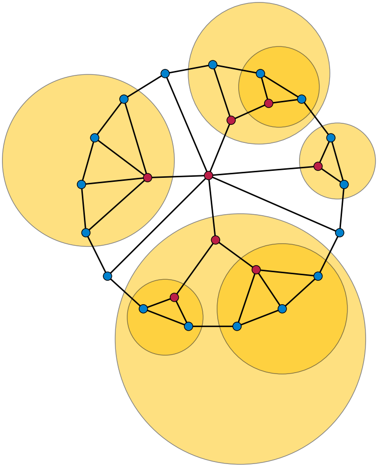
Output: a drawing with

- Clusters = Jordan curves
- No two cluster curves cross
- No two edges cross
- Edge can only cross curve (once) when it has one endpoint in the cluster

Polynomial when all clusters connected, many other special cases
[Lengauer 1989; Feng et al. 1995; Cortese and Di Battista 2005]

Open: What is the complexity of testing clustered planarity?

Warmup: Polyhedra from clustered cycles



Hierarchically cluster a cycle with all clusters = paths + one cluster for whole cycle

Augment cycle to *cluster graph* with one vertex per cluster, adjacent to sub-cluster vertices and to cycle vertices that are not in sub-clusters

Result is a *Halin graph* (tree + cycle through leaves)

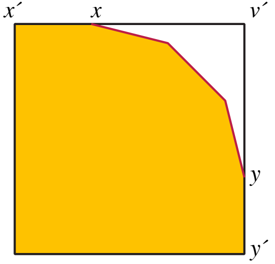
Realizable as polyhedron with cycle = base face

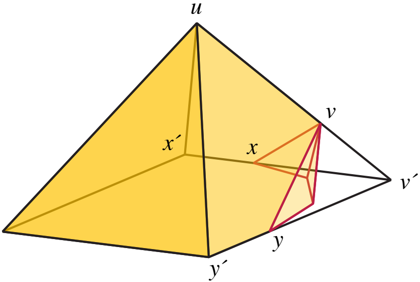
Inductive realization of Halin graphs

Special case of Steinitz but has easier proof

Induction on # clusters:

Collapse a cluster to a vertex, realize inductively, uncollapse

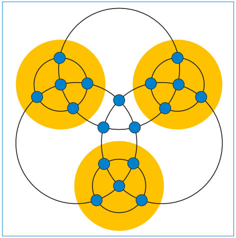


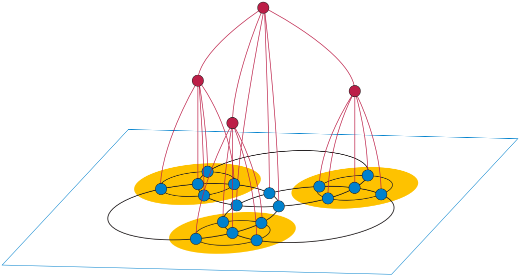


Uncollapse = replace base polygon vertex by convex chain

Open: How small can integer coordinates be?

Treetopes from clustered planar graphs





Theorem: Treetope = cluster graph of a clustering such that:

Underlying planar graph (base facet of treetope) is 3-connected

Collapsing any cluster or complement gives a 3-connected minor

Each cluster vertex has degree at least four

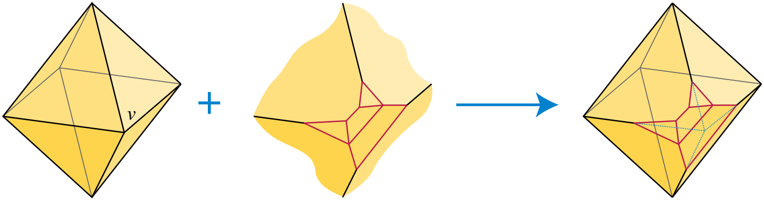
At most one edge connects each two disjoint clusters,
complements or single vertices, unless they cover whole graph

Inductive realization of treetopes

Same basic idea as Halin graph realization

Induction on # clusters:

Collapse a cluster to a vertex, realize inductively, uncollapse



Uncollapse = replace base polyhedron vertex by polyhedral surface

To find the surface, use (polar version of) the result that any 3d polyhedron can be realized with one face shape specified

[Barnette and Grünbaum 1970]

Polynomial time recognition of treetopes

Repeat:

- Find a vertex v that looks like a cluster vertex
  - At least four neighbors
  - Its neighborhood = planar graph + isolated vertex
  - No two neighbors adjacent to same non-neighbor
  - Delete it and contract non-neighbors $\Rightarrow$ 3-connected
  - Not marked as part of base polyhedron
- Contract v and its neighbors
- Mark contracted vertex as part of base

Verify that this process reduces to polyhedron + universal vertex

(Selected vertex might not be a cluster vertex, but if it is not, the contraction gives the same result as if we had found the cluster vertex for its cluster.)

Conclusions

Three-dimensional convex polyhedra:

Exact characterization known

Still many unsolved questions re coordinate size

Three-dimensional non-convex polyhedra:

Only special cases for topological spheres known

Four-dimensional convex polytopes:

Only special cases (simple, treetopes) known

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