

YOUNG MATHEMATICIANS AT WORK
Constructing Fractions, Decimals, and Percents

*Minilessons for Operations
with Fractions, Decimals,
and Percents, Grades 5–8*

FACILITATOR'S GUIDE

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Overview

Minilessons are typically highly focused and short (ten or fifteen minutes in duration). Their design—a structured series of computation problems or strings—is used to build and highlight number relationships and operations and to develop efficient mental-math computation. To this end, they are guided and explicit in nature.

There are two CD-ROMs in this set of materials on minilessons. They are both on operations with rational numbers. One is on addition and subtraction (with video from sixth-grade classrooms), the other on multiplication and division (with video from seventh- and eighth-grade classrooms).

With these CD-ROMs, participants in your workshops can examine each minilesson's design and think about how the choice of number might be used to support the development of children's strategies for computation. Participants are also given opportunities to think about how the clock, double number line, and array can be used both as models to represent students' thinking and as tools for students to think with.

As participants work with the CD-ROMs, they will learn to

- listen to and accurately describe students' strategies;
- analyze students' strategies and think about the big ideas underlying these;
- represent students' strategies using a variety of models; and
- design their own minilessons.

Journey 2

Addition and Subtraction of Fractions, Decimals, and Percents

Journey 1 was designed to help participants think about the important role minilessons play in the development of computational efficiency. To support participants, *Journey 1* focused on (1) developing and deepening their own content knowledge and mental-math strategies; and (2) using the minilessons provided on the CD-ROM as a tool to hone their powers of observation and analysis.

But these are beginning steps. To effectively use minilessons in their own practice, participants also need to know *how* to represent children's strategies as they unfold and have a pedagogical repertoire that supports individual student development and sustains the mathematical life of an entire classroom community. Although these are sophisticated ideas and not easily or quickly developed, they are critical next steps in participants' journeys.

Journey 2 has been designed toward this end. Here participants revisit the minilessons they have previously analyzed in *Journey 1*,³ but this time the focus shifts from analyzing student strategies to thinking about what a *teacher* does to affect and support learning. As part of their work in *Journey 2*, participants are given opportunities to:

- model student strategies and compare their representations with those of the classroom teacher in the video clip;
- think about the developmental progression of models (how models evolve from tools to represent thinking into models *for* thinking); and
- practice designing their own strings.

THE CLOCK MODEL

The Clock Model

⊙ *Josi used twelfths when she solved $1/2 + 1/6$. Joel asks the students where the twelfths show up on the clock. Describe your noticings.*

Because models and modeling play such an important part in the development of mathematical thinking, participants ought to have a clear understanding of how they

³There are, however, several exceptions in *Journey 2: Multiplication and Division of Fractions, Decimals, and Percents*. There, participants are given some strings (or excerpts from strings) that they have not previously analyzed.

are developed and how they can be used effectively in minilessons to support learning. As part of their experiences in *Journey 1*, participants were given an opportunity to think about modeling as they analyzed each of the minilessons. But that was only the beginning of participant work with the models. In order for them to use the models effectively in their own practice, they first need to understand how models are developed and the different ways they can be used as tools to support and represent student strategies.

Two of the folders in *Journey 2*, “The Clock Model” and “The Double Number Line,” have been designed toward this end. As participants work with these folders, they can explore a number of important questions connected to models and modeling. These include:

1. What are the different uses of these models?
2. How are students’ developmental needs supported by the different uses of these models?
3. Is there a developmental progression for these models? Are some uses of the model precursors to others?
4. What role does the model play in helping students visualize different strategies?
5. How might one represent a given strategy with each model?

To think about the developmental progression of each model, participants must first be able to clearly delineate the ways it might be used. The clips on the pages in each folder can be used to help participants reflect on how students’ developmental needs determine specific teaching choices. As participants reflect on Joel’s teaching choices, they will also have to consider how models can be used not only to support the development of big ideas and landmark strategies, but also to help students build a network of mathematical relationships.

In order for participants to use the clock effectively as a tool to represent student strategies, they need to understand several key ideas about the development and use of this model. Participants need to consider:

1. why, even though specific fractions may come up quite naturally in relationship to the clock model, understanding these relationships may not be automatic for students;
2. how they might use the clock model to support the development of these relationships (e.g., how different fractions are equivalent); and
3. what the starting place might be for this kind of discussion (i.e., what ideas might be the most accessible to students).

On the CD-ROM page *The Clock Model*, Joel explores twelfths on the clock with his students. But what mathematical reasons might he have had for doing this? To understand the significance of this choice, participants need to think about how students might know there are twelve numbers on an analog clock (this is, after all, social knowledge) and that each numeral on the clock represents five minutes without necessarily understanding that these markings also represent twelfths or what these

FACILITATION TIP 30



In preparation for their work in *Journey 2*, you might ask participants to visit (or revisit) the landscape of learning¹ of the big ideas, strategies, and models comprising fractions, decimals, and percents. Although participants in *Journey 1* have had many different experiences with the big ideas, strategies, and models, these experiences may not necessarily have coalesced into a coherent model of learning they can draw from as they think about analyzing and creating strings of their own or how they would represent student strategies.

¹See Figure 8.5, pp. 136–137, in the companion book, *Young Mathematicians at Work: Constructing Fractions, Decimals, and Percents*.

FACILITATION TIP 31



Understanding the power of models and modeling is a critical part of participants’ journeys. As part of this work, participants need to develop a solid understanding of how the clock model can be used didactically not only to represent student solutions, but also to develop and explore big ideas underlying their strategies. Without this knowledge, participants will have a difficult time using the clock model effectively in their own practice.

Because participants have previously viewed the clips on this page (see Joel’s minilesson, Folder 1, *Journey 1*), they may wonder why it is necessary to rewatch them. If comments like this arise, be sure to help participants understand how in this viewing the lens has shifted from exploring the mathematical ideas underlying student strategies to thinking about how the clock can be used didactically as a tool both to represent student solutions and to develop and explore big ideas underlying their strategies.

twelfths mean in relation to an hour (i.e., because the unit of measure on the clock is the hour; five minutes is $5/60$ [or $1/12$] of an hour). As part of this analysis, participants also need to think about:

1. what mathematical ideas—if a child understands that the numerical markings on a clock can be interpreted as twelfths—underlie this mental construction; and
2. how these ideas might be brought up in whole-group discussion in a way that helps other students construct them.

FACILITATION TIP 32



Why might the relationship between the numbers on an analog clock and the fractional part of an hour be a difficult one for students? One reason is that in order for students to understand how the numbers on the clock represent twelfths they need to shift the unit of time they are working with. Keeping the whole in mind when considering a fraction is a big idea in the development of proportional reasoning, and enormously difficult for students to do.¹ This is because to do this students have to think about the whole (one hour) and the relationship of each marking on an analog clock to that whole. From this perspective, the “6” on the clock has two meanings: it does not just mean 6 out of 12 markings (or half of the markings on the clock), but 6 groups of five-minute chunks that are ultimately also related to the unit of time measured on a clock (i.e., $6 \times 5/60$). Thus, understanding that all the numerals on the clock have two meanings is a critical idea for students to develop. For this reason, Joel spends time in his minilesson exploring twelfths.

¹For more information on unitizing, refer to Susan J. Lamon’s book, *Teaching Fractions and Ratios for Understanding*, which is listed under INFO on the menu bar (see Background Reading).

In Clip 15 Joel refers to a previous strategy used by Cory,⁴ where she thought of the answer to the problem $1/3 + 1/4$ in terms of twelfths (i.e., when she simplifies her answer $35/60$ to $7/12$). In response to Joel’s highlighting of Cory’s strategy, Josi uses equivalence to turn $1/2 + 1/6$ into $6/12 + 2/12$. Because Josi’s solution utilizes a number of important mathematical ideas, Joel spends time examining her strategy with the other students.

To do this and to help make the mathematical ideas in Josi’s strategy explicit, Joel asks in Clip 16, “where do these twelfths show up on the clock?” As Suzie explains her thinking about how the markings on the clock are related to twelfths, Joel models her ideas on a “clock” (i.e., a circle he has drawn on the board to represent a clock), indicating how each number not only equals five minutes, but also can be thought of as $1/12$ of an hour. Joel describes them as “12 five-minute chunks,” and to support this idea indicates both $1/12$ and 5 minutes on the appropriate portion of the clock. Because this is so important, Joel uses paraphrasing and pair talk to have his students further explore this idea.

In Clip 17, Joel’s question, “What makes these fractions make sense . . . how does the clock model help us?” pushes students to think about how the clock model helps them make sense of certain fractions. Five important ideas arise as part of this discussion:

In Clip 17, Joel’s question, “What makes these fractions make sense . . . how does the clock model help us?” pushes students to think about how the clock model helps them make sense of certain fractions. Five important ideas arise as part of this discussion:

1. how the clock (the context of time) is a way to make sense of different fractions; one can think about them as being “part of an hour” (in Clip 17 Alexa refers to the context—the minutes it takes “you [Joel] to jog, walk, run”—and how within this context the amount of time for each activity can be understood as a portion of time or “minutes out of an hour”);
2. how some units of time are equivalent, i.e., $30/60 = 6/12 = 3/6$ (in Clip 17 Veronica discusses the clock in relationship to equivalence, “Each one that makes up one, when the denominator becomes 12, it makes sense because there are 12 spaces [markings] on the clock and each one is five minutes . . . each one [marking] that makes up the whole number on the numerator can equal one space on the clock”);
3. how fractions are related to division (in Clip 17, Alexa explores how $40/60$ is related to $8/12$; “we’re dividing the 60 into 12 and each $1/12$ is five minutes [because 5×12 is 60] and we’re dealing with $8/12$; it equals 40 minutes [because 5×8 is 40]”);

⁴The strategy referred to by Joel was actually used by Lucy (see *Journey 1: Addition and Subtraction of Fractions, Decimals, and Percents*, “Strings: An Example,” *Joel’s String*, Clip 4).

4. how one can use one's knowledge of factors and multiples to think about the ways in which fractions are related (in Clip 18, Lucy talks about why 12 and 60 are good numbers for working with fractions: "because a lot of numbers go into them; there are a lot of factors . . ."); and
5. how the clock itself has specific divisions (i.e., units) that can help one think about different fractions (in Clip 18, Josi says, "you can see the clock broken up into twelfths, sixtieths, and sixths").

Fractions on the Clock

- ☉ *With which fractions might the clock model be helpful?*

Before viewing children's ideas on how and when the clock model might be helpful, it is important for participants to reflect on this question themselves. As part of any discussion, be sure to help participants think about how

- some landmark fractions are natural within this context because they are part of the common language of time (e.g., quarter of an hour or half an hour) while others (like thirds) are not;
- the markings on a clock can help students make sense of different fractions (e.g., how the minute marking represents $1/60$ of an hour; how the numbers on an analog clock represent twelfths of an hour, etc.);
- the relationship between units might be used to help students think about equivalence (e.g., $1/6$ of an hour is equal to $2/12$); and
- less common kinds of fractions (120ths, 30ths) might come up for discussion using this model.

As part of this discussion it might also be helpful to have participants think about what other kinds of landmark fractions might come up (e.g., fifths), how these might not be as easy to model with the clock, and how these kinds of fractions might push students to begin to use the clock as a model *for* thinking.

Students Discussing Fractions on the Clock

- ☉ *View the children discussing how and when the clock model can be helpful.*

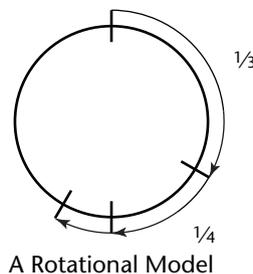
In Clip 19, Joel asks his students to think about what "fractions . . . would make sense to work with a clock if we know that a clock is 60 minutes and we've got all this other stuff that we've talked about?" Students easily answer this question and Joel lists their answers on the board ($1/3$, $1/2$, $1/4$, $1/6$, $1/12$, $1/60$). The response, $1/120$, however, needs to be explored with the students not only because this is the first time they have moved away from the landmark fractions and units of time examined in the minilesson, but because the relationship of $1/120$ to $1/60$ may not be apparent to some children in the class. If $1/60$ means one minute out of an hour, what relationship does $1/120$ have to this? Because Joel recognizes that this relationship may be problematic for some of his students, he asks, "why would you say [suggest] that?" Lucy responds, "I think because that's half of the 60th. And I think that if you had

FACILITATION TIP 33

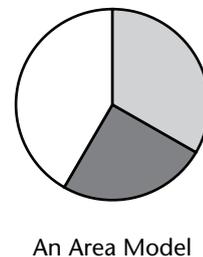


There are two ways to model strategies with the clock. One is a rotational model (where the fraction is represented by how far the hand has moved around the clock face); the other is an area model (where the fraction is represented as a portion of the clock). Similar strategies for the problem $1/3 + 1/4$ could be represented differently.

$$1/3 + 1/4 = 20/60 + 10/60 + 5/60$$



$$1/3 + 1/4 = 20/60 + 15/60$$



something with that denominator you would just know that one of those [1/120] would be 30 seconds because 1/60 is a minute and . . .”

After creating this list, Joel asks his students what they notice about their list, what makes these fractions work with a clock model. The first response he gets, “they’re equivalent to each other and to 60,” indicates that there may still be some confusion in the class about the ways in which these different fractions are related on the clock. As Joel pushes his students to think about this further they begin to talk about the multiplicative relationships between the denominators, and the fact that, as Lucy says, “all of the denominators are either factors or multiples of 60.”

To end the minilesson, Joel not only paraphrases how the fractions on their list are related, but links these ideas back to the computation they had previously done in the minilesson. He says, “as far as a clock model is concerned, these fractions all make sense to work with . . . the clock models help us think about adding the fractions because all of their denominators are either factors of 60 and there’s 60 minutes in an hour or even . . . multiples of 60. So we can think of them as chunks of 60 minutes really easily and the clock model helps us to add these fractions together.”

FACILITATION TIP 34



Part of student confusion with the relationship of 1/120 to 1/60 arises from some common misconceptions children hold about fractions: that the larger the denominator the larger the fraction. This occurs because children often apply their knowledge of operations with whole numbers to fractions. If 120 is double 60, why isn’t 1/120 double 1/60? Joel has Lucy explain her choice of 1/120 for their list to make these ideas explicit for her peers. If 1/60 means one minute then 1/120 is half that amount of time, or 30 seconds.

What Would You Do Next?

☉ *Joel’s minilesson consists of the following problems:*

$$1/3 + 1/4$$

$$1/2 + 2/3$$

$$10/60 + 1/2$$

$$1/3 + 25/60 + 1/4$$

$$1/6 + 1/2$$

Write some more addition of fraction strings to use with a clock model.

This page has been built into the CD-ROM to give participants an opportunity to begin to construct the tools necessary for writing strings (this will be explored again in depth in the third folder in *Journey 2: Addition and Subtraction of Fractions, Decimals, and Percents*, “Developing Strings”).⁵ Although participants may not think they need this experience—indeed sometimes comments arise such as, “Just give us a book of strings to use as minilessons”—there are several reasons why writing strings is critical to participants’ development. To understand these one needs to examine the tools participants will use as they create their own strings.

First, the starting place for writing fraction strings that can be used with the clock model is the landscape of learning. In order to carefully craft a set of problems in a minilesson, participants need to be crystal clear about what the big ideas and strategies are on the landscape of learning and understand how they are interrelated. Second, as participants use the landscape of learning to design strings to support the development of specific strategies, they also need to keep in mind some key questions:

1. What *other* potential strategies might students use?
2. How are the different student strategies related?

⁵For sample strings to use with the clock model, refer to Appendix A.



Put a problem like $2/12 + 3/6$ on the board. Ask participants to create the next problem in the string and share with the whole group the problems they have created as possible next steps and the mathematical reasons for their choices. You might continue writing the string as a whole group or ask them to continue the string on their own as an assignment. When they turn this in, analyze their work for (1) how they structure the string; (2) what strategies or big ideas they think their problems are developing; (3) what strategies they anticipate students using given the problems in their string; and (4) how they might represent these.

ACTIVITY 11 Writing Strings 2



Ask participants to create their own strings to be used with the clock model. As part of this assignment, ask them to keep a record of the potential strategies their strings might elicit and how they will represent these strategies. Have participants do their strings with a small or large group and record the strategies that the group offered as solutions to the problems in their strings. When they have finished their strings ask them to reflect on the following questions:

1. Did you anticipate the strategies that were offered as solutions to the problems in your string?
 2. Did any of the strategies confuse you? If so, why?
 3. How did you probe strategies you did not understand? What questions did you ask? When and why did you ask them?
 4. How did you represent different strategies? What model(s) did you use?
 5. If you do this string again, how would you modify it? Why would you do this?
-

3. How might these strategies be represented?
4. How might connections be built between different strategies?
5. What might students' struggles be?

The tools needed to answer these questions are the very ones participants have been developing in their CD-ROM work. The difference now is that participants will be using what they know to *envision* how a string they develop might play out in a minilesson. This experience is essential for participants to have if they are to successfully implement minilessons in their own practice.

It is important that participants internalize the questions delineated above into their idea of what it means to *write a string* because it is exactly the questions they ask themselves in creating a string—how well they anticipate what *might happen*—that will prepare them for what *does happen*. The idea that developing strings is not just about creating a set of problems, but about anticipating how these problems might be solved and represented cannot be given to participants in a book of strings. These ideas can be constructed only in the act of *doing* (writing strings, planning what kinds of strategies might come up for discussion, etc.) and *reflecting* (thinking about how effective one's plan was in light of what actually happened). This page is designed, then, to provide participants with these kinds of experiences.

FACILITATION TIP 35


After participants have an opportunity to write strings as follow-ups to Joel's minilesson, it is important to give them an opportunity to actually do their strings (see Activities 10 through 13). Doing so will help participants think about a number of critical ideas they will need to develop if they are to effectively use strings in their own practice. Toward this end, it will be helpful to participants to reflect on the following questions after they have done their string:

1. Did you anticipate all strategies that were offered when you planned your minilesson?
2. Were any strategies difficult to understand? Represent?
3. What role did questioning play in helping you probe or understand the strategies offered?
4. How did you model different strategies?
5. How flexible was your minilesson? Did you do your string as written or alter it in the process of doing the minilesson? What affected your decision making?

The last question is especially important for participants to consider because it is not uncommon for beginners to think that the problems in a string are written in stone. This misconception can lead to disastrous results when the students in their class fail to respond on cue to the expectations they have for their string. That a string can be altered midcourse depending on the strategies being offered is an idea participants need to entertain if they are to use strings as tools for developing thinking. Although doing so is enormously difficult for beginners because it means letting go of the "script" and improvising, participants still need to be exposed to this idea.

FACILITATION TIP 36


Often participants assume that the major challenge in creating an effective minilesson is writing a good string. The assumption is that a well-written string somehow translates into a well-taught lesson. Participants are understandably taken aback when they do their well-written string with a group and find that this is not as easily managed as they had anticipated. The disequilibrium they experience here can be one of the building blocks to support their development. At the heart of this discussion is the question, "What tools—both mathematical and pedagogical—does one need for doing an effective string?"

Although it may appear that there are many possible choices for the strings participants design as possible next steps, most of what they write will fall into the following categories of developing strings:

- to continue to work with equivalence and landmark fractions;
- to examine, with three or more addends, effective methods for pairing problems to make adding easier (e.g., with the problems $2/6 + 1/4 + 1/3 + 5/6$, using the associative property of addition to change the problem into $[(2/6 + 5/6) + 1/4 + 1/3]$);
- to support making a whole, e.g., $2/3 + 1/4 + 2/3 = 2/3 + [1/3 + 1/3] + 1/4 = [3/3 + 1/3] + 1/4 = 1 + 1/3 + 1/4$; and
- to work with fractions that are factors or multiples of sixty, but that are not as easy to model with a clock (fifths, twentieths, etc.).

FACILITATION TIP 37

As participants set off to write their own strings, keep track of the *kinds* of problems they create and their mathematical reasons for doing so. Some participants, who may hold a behaviorist approach to learning where students' mastery of a skill (in this case a desired strategy) is based on a considerable amount of practice, may create problems that they think will accomplish this goal.

Other participants may create exactly the same kinds of strings, but for reasons that are rooted in a rather superficial understanding of how learners construct knowledge. Within this framework, participants might think that doing string work with students is a way to help them *see* the relationships, as if the relationships built into the string can exist *outside* of the students' mental actions on the problems.

In both instances, albeit with different intentions, participants may create problems that spoon-feed the relationship they are trying to help the students to "get." Often they create pairs of problems with a narrow range (e.g., $1/4 + 1/4$, $2/4 + 1/4$, $3/4 + 1/4$, and so on) as if limiting student strategies is a way to control thinking and obtain the desired results (e.g., the development of a specific strategy).

Some participants may believe that it is better to work with *easier problems* until students *master* the idea that is being developed in their string. This kind of structuring can be connected to the idea that learning takes place in small steps that need to be carefully scaffolded to provide students with enough supports to help them "master" the "skill" they are working on. For example, participants may think that it is better to start with "easier" numbers until students *have* the idea and then to move to "harder" numbers (e.g., design strings to help students *master* problems with common denominators before moving on to problems with different denominators).

Whatever strings participants create, it is important to challenge them to think about how their strings might both support and stretch student thinking, and develop efficient computation strategies. As part of this discussion, it might be helpful to return to the strings on the CD-ROM and ask participants to reexamine the structure of each string. Here, it is important for them to think about the duality of the problems in these strings: how they are designed to work within the developmental range of students and, at the same time, push thinking. Minilessons that effectively do this are founded on a constructive view of learning—the exact idea that participants will need to reconsider if they are to learn how to create strings that support development.

FACILITATION TIP 38

A common misconception held by novices is that good listening skills are key to being able to represent students' thinking. Often they experience a considerable amount of disequilibrium as they attempt to represent students' thinking because good listening skills, although important, are only one part of the picture. Being able to represent student strategies is also contingent on

- having the content knowledge to understand the mathematical ideas in a strategy *as it unfolds in real time*;
- being able to ask questions to probe or clarify thinking; and
- having sufficient knowledge of models and modeling to accurately represent a strategy.

The cohesion of all these seemingly disparate elements is a significant part of the work done in *Journey 2*.

There are many different strings participants can write. To help them connect their strings to what has happened previously, the following questions might be helpful:

1. What strategies is your string designed to support?
2. What might students' potential solutions be?
3. How would you represent these solutions with the clock?
4. How does your string build upon the strategies you saw students using?
5. How is the choice of problems in your string related?

In order to write effective strings, participants need to understand

- how the structure of a string influences student strategies;
- how the design of a string builds mathematically in a way that helps students make connections between problems and use previous solutions to help them with new problems; and
- how modeling has the potential to deepen student thought and affect strategies with other problems in the string.

FACILITATION TIP 39



Because it is sometimes difficult for novices to write strings, participants may need a lot of support as they attempt to do so. As participants design their strings, look for the following:

1. Where do they start their string? It is not uncommon for participants to begin their string with the most difficult problem. They may do this because they may think it is important to challenge students right from the start, or they may want to collect a wide range of strategies from them so that there will be a rich conversation immediately. Should participants do this, it may be helpful to redirect them to the structure of different strings they have analyzed on the CD-ROM. Ask them to think about where the teachers began their strings and why they might have done this.
2. What strategies do participants anticipate students using as they do the string? Some participants may create potentially effective strings, but have difficulty imagining strategies other than the ones they have designed the string to bring up. Because anticipating a wide range of student strategies and knowing how to represent these are key elements of doing strings effectively, it is important to push participants to think of what other potential strategies their string might elicit. One way to bring this up for discussion is to ask participants, as they do their strings with small or large groups, to record the different strategies that arise and then compare what they anticipated as strategies with what actually occurred (see Activity 11: "Writing Strings 2"). Not only can rich discussions be generated from such analyses, it is also one way to develop participants' ability to anticipate the strategies that their string might produce.
3. How do participants plan to represent different strategies? Knowing how to represent a strategy is a critical piece of being able to effectively use strings to support mathematical thinking. If participants struggle with how to model various strategies, it might be helpful for them to revisit different strings on the CD-ROM and think about how Joel used the clock to model student thinking.



Just as the best way to learn how to write strings is to have many experiences designing them, the best way to learn how to do strings is to actually do them on a regular basis. But even then, doing strings regularly is no guarantee that one will learn to do them effectively. Here, a third factor is critical: self-reflection and the ability to use those evaluations as the basis for change. To support participants in the development of this kind of self-reflection, it is important to help them *hear* the questions they need to ask themselves. These include

1. Was my string effective? What made it so?
2. Did the mathematical ideas and strategies I designed my string around actually come up for discussion?
3. Did I adequately plan for *all* the strategies that occurred? If not, which strategies were not anticipated? What was it about the choice of numbers in my string that might have made this strategy (or these strategies) come up for discussion?
4. How prepared was I with questions to probe or clarify a strategy?
5. Was I able to use other pedagogical tools like paraphrasing, pair talk, wait time, etc.? What might be my areas of strength and what might I need to improve?

FACILITATION TIP 41



Because this kind of self-reflection is not easily developed, it is important for participants not only to have opportunities to do strings they have created (it is crucial here that their minilessons be *carefully* planned to anticipate potential strategies, ways to represent them, etc.) but also to get feedback on their minilesson by an experienced observer who can direct their thinking to areas they might need to consider. As participants do their minilessons, keep the following in mind:

- Do they listen to participants' strategies? Do they understand a strategy? Do they miss the mathematical ideas?
- Do they use questions to probe or clarify thinking?
- How accurate are their representations? Do they use the array to model strategies?
- Do they pace their string to develop key mathematical ideas? For example, do they spend more time eliciting strategies for pivotal problems or treat each problem with the same weight?
- Do they have the pedagogical tools to involve everyone in the conversation? Do they use wait time or paraphrasing?

Be sure to give participants feedback on their minilessons. This can be done in two main ways. One is to interact with the participant as the string is being done. Here it is easier to model the kinds of things (e.g., specific pedagogical tools like paraphrasing or wait time; how to use questions to probe a strategy; how to build connections between strategies) they need to develop. The other is to watch a minilesson, take notes, and have a post-string conference in which your observations are shared with each individual. The latter, although time-consuming, may be necessary for participants who have no experience with a coteaching model and find it difficult to be interrupted in the act of teaching.¹

¹It is important to give participants a choice here because if they are easily embarrassed or threatened by their lack of know-how, this could have disastrous consequences in terms of their being willing to take other kinds of risks in front of their peers.

TECH TIP 2

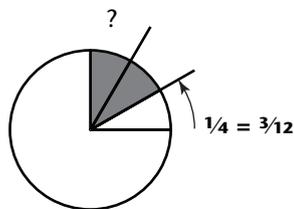


For information on how to use the Drawing Tool, see the Help file located in the TOOLS menu.

$1/4 - 1/12$

- ⊙ To the right of the Notepad is a Palette. When you click on it you will get a Drawing Tool. Draw a picture of the clock to solve the problem $1/4 - 1/12$.

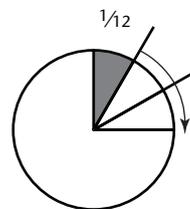
Up until this point in the CD-ROM, participants' use of the clock model has been in relation to addition of fractions. Now, to help participants to think more deeply about this model, they are asked to consider how they would use the clock to represent their solution to the subtraction problem, $1/4 - 1/12$. Whatever participants' strategies, be sure to explore the underlying mathematical ideas. For example, some participants may think of $1/4 - 1/12$ in terms of removal; others might think of it in terms of a missing addend (see the representations).



A removal strategy modeled on the clock

$$1/4 - 1/12 = ?$$

$$3/12 - 1/12 = 2/12$$



A missing addend strategy modeled on the clock

$$1/12 + ? = 1/4$$

$$1/12 + 2/12 = 3/12$$

Whatever participants' solutions, be sure to explore how accurately their representations of their strategies model the mathematical ideas they are using. For example, participants may say, "I knew 15 minutes minus 5 minutes was 10 minutes," and model this solution with the clock as shown in the first example above. What needs to be explored here is the connection of 15 minutes and 5 minutes to the original problem and what mathematical ideas they used (however quickly they did this mentally) to solve the problem $1/4 - 1/12$.

The idea here is to help participants think about how the clock model can be used as a tool to make mathematical ideas explicit (e.g., 15 minutes is equivalent to $3/12$ of an hour) that may be implicit in a given strategy. Pushing participants to reflect on why it is important to explore the mathematical meanings behind strategies and how the clock model (or any model) is a tool with which to do this will support the development of their own visions of modeling. Ultimately, these experiences will influence how participants will use the clock model in their own practice.

Backburner: The Clock Model

- ⊙ This is the last page in the folder "The Clock Model." However, you may have other questions on this topic that you would like to investigate. Go to the TOOLS menu above and add them to your Backburner notes.

See Backburner notes on p. 11.

THE DOUBLE NUMBER LINE

Looking Back

- ⊙ Look back at some of the problems in the strings you have seen so far. Joel and Michael made use of the double number line in their minilessons. Describe the way they used the double number line.



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