On Travelling Out of Shire

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Abstract. The abstract should summarize the contents of the paper using at least 70 and at most 150 words: We present a way how Hobbits could travel. We describe all steps...

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1 Introduction

The Hobbits[1] do not travel often therefore if they decide to... [1, page 21–23]

2 First steps

At the beginning you should get a map of all the areas you wish to go through. On this map you should clearly mark where you are and where your destination is. Figure 1 shows an example of such map. Take a good bag [2] with you.

Note 1. We have borrowed the following text from the LLNCS demonstration file to show very useful elements of LNCS LaTeX2e class and the beauty of mathematics.

3 More Examples

With this chapter, the preliminaries are over and we will give an equation:

$$\begin{aligned} \dot{x} &= JH'(t,x) \\ x(0) &= x(T) \end{aligned}$$

with $H(t,\cdot)$ a convex function of x, going to $+\infty$ when $\|x\| \to \infty$.

In this section, we will consider the case when the Hamiltonian $H(x)$ is autonomous. For the sake of simplicity, we shall also assume that it is C^1.

We shall first consider the question of nontriviality, within the general framework of (A_∞, B_∞)-subquadratic Hamiltonians. In the second subsection, we shall look into the special case when H is $(0, b_\infty)$-subquadratic, and we shall try to derive additional information.

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1 Hobbits are an unobtrusive but very ancient people, more numerous formerly than they are today; for they love peace...

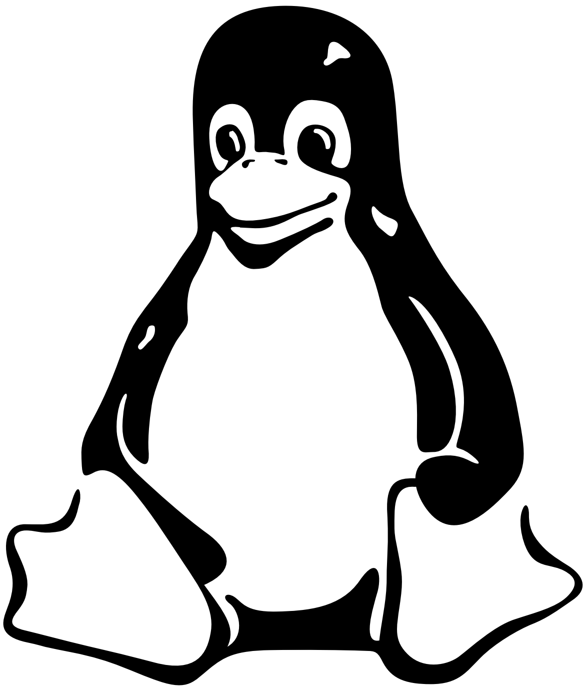

Figure 1. Penguin, in contrast to hobbits, likes mathematics and can travel even without a good map.

The General Case: Nontriviality. We assume that H is (A_∞, B_∞)-subquadratic at infinity, for some constant symmetric matrices A_∞ and B_∞, with $B_\infty - A_\infty$ positive definite. Set:

$$\gamma := \text{smallest eigenvalue of} \quad B_\infty - A_\infty \tag{1}$$

$$\lambda := \text{largest negative eigenvalue of} \quad J\frac{d}{dt} + A_\infty \ . \tag{2}$$

Theorem 5 tells us that if $\lambda + \gamma < 0$, the boundary-value problem:

$$\begin{aligned} \dot{x} &= JH'(x) \\ x(0) &= x(T) \end{aligned} \tag{3}$$

has at least one solution $\overline{x}$, which is found by minimizing the dual action functional:

$$\psi(u) = \int_o^T \left[\frac{1}{2}\left(\Lambda_o^{-1}u, u\right) + N^*(-u)\right] dt \tag{4}$$

on the range of Λ, which is a subspace $R(\Lambda)_L^2$ with finite codimension. Here

$$N(x) := H(x) - \frac{1}{2}\left(A_\infty x, x\right) \tag{5}$$

is a convex function, and

$$N(x) \le \frac{1}{2}\left(\left(B_\infty - A_\infty\right)x, x\right) + c \quad \forall x \ . \tag{6}$$

Proposition 2. *Assume $H'(0) = 0$ and $H(0) = 0$. Set:*

$$\delta := \liminf_{x \to 0} 2N(x) \|x\|^{-2} \ . \tag{7}$$

If $\gamma < -\lambda < \delta$, the solution $\overline{u}$ is non-zero:

$$\overline{x}(t) \neq 0 \quad \forall t \ . \tag{8}$$

Proof. Condition (7) means that, for every $\delta' > \delta$, there is some $\varepsilon > 0$ such that

$$\|x\| \leq \varepsilon \Rightarrow N(x) \leq \frac{\delta'}{2} \|x\|^2 \ . \tag{9}$$

It is an exercise in convex analysis, into which we shall not go, to show that this implies that there is an $\eta > 0$ such that

$$f \|x\| \leq \eta \Rightarrow N^*(y) \leq \frac{1}{2\delta'} \|y\|^2 \ . \tag{10}$$

Figure 2. This is the caption of the figure displaying a white eagle and a white horse on a snow field

Since u_1 is a smooth function, we will have $\|hu_1\|_\infty \leq \eta$ for h small enough, and inequality (10) will hold, yielding thereby:

$$\psi(hu_1) \leq \frac{h^2}{2} \frac{1}{\lambda} \|u_1\|_2^2 + \frac{h^2}{2} \frac{1}{\delta'} \|u_1\|^2 \ . \tag{11}$$

If we choose δ' close enough to δ, the quantity $\left(\frac{1}{\lambda} + \frac{1}{\delta'}\right)$ will be negative, and we end up with

$$\psi(hu_1) < 0 \qquad \text{for} \ \ h \neq 0 \ \ \text{small} \ . \tag{12}$$

On the other hand, we check directly that $\psi(0) = 0$. This shows that 0 cannot be a minimizer of ψ, not even a local one. So $\overline{u} \neq 0$ and $\overline{u} \neq \Lambda_o^{-1}(0) = 0$. □

Corollary 3. *Assume H is C^2 and (a_∞, b_∞)-subquadratic at infinity. Let $\xi_1, \ldots, \xi_N$ be the equilibria, that is, the solutions of $H'(\xi) = 0$. Denote by ω_k the smallest eigenvalue of $H''(\xi_k)$, and set:*

$$\omega := \mathrm{Min} \left\{\omega_1, \ldots, \omega_k\right\} \ . \tag{13}$$

If:

$$\frac{T}{2\pi} b_\infty < -E \left[-\frac{T}{2\pi} a_\infty\right] < \frac{T}{2\pi} \omega \tag{14}$$

then minimization of ψ yields a non-constant T-periodic solution $\overline{x}$.

We recall once more that by the integer part $E[\alpha]$ of $\alpha \in \mathbb{R}$, we mean the $a \in \mathbb{Z}$ such that $a < \alpha \leq a+1$. For instance, if we take $a_\infty = 0$, Corollary 2 tells us that $\overline{x}$ exists and is non-constant provided that:

$$\frac{T}{2\pi} b_\infty < 1 < \frac{T}{2\pi} \tag{15}$$

or

$$T \in \left(\frac{2\pi}{\omega}, \frac{2\pi}{b_\infty} \right) \ . \tag{16}$$

Proof. The spectrum of Λ is $\frac{2\pi}{T}\mathbb{Z} + a_\infty$. The largest negative eigenvalue λ is given by $\frac{2\pi}{T}k_o + a_\infty$, where

$$\frac{2\pi}{T}k_o + a_\infty < 0 \leq \frac{2\pi}{T}(k_o+1) + a_\infty \ . \tag{17}$$

Hence:

$$k_o = E\left[-\frac{T}{2\pi} a_\infty \right] \ . \tag{18}$$

The condition $\gamma < -\lambda < \delta$ now becomes:

$$b_\infty - a_\infty < -\frac{2\pi}{T}k_o - a_\infty < \omega - a_\infty \tag{19}$$

which is precisely condition (14). □

Lemma 4. *Assume that H is C^2 on $\mathbb{R}^{2n} \setminus \{0\}$ and that $H''(x)$ is non-degenerate for any $x \neq 0$. Then any local minimizer $\widetilde{x}$ of ψ has minimal period T.*

Proof. We know that $\widetilde{x}$, or $\widetilde{x} + \xi$ for some constant $\xi \in \mathbb{R}^{2n}$, is a T-periodic solution of the Hamiltonian system:

$$\dot{x} = JH'(x) \ . \tag{20}$$

There is no loss of generality in taking $\xi = 0$. So $\psi(x) \geq \psi(\widetilde{x})$ for all $\widetilde{x}$ in some neighbourhood of x in $W^{1,2}\left(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n}\right)$.

But this index is precisely the index $i_T(\widetilde{x})$ of the T-periodic solution $\widetilde{x}$ over the interval $(0,T)$, as defined in Sect. 2.6. So

$$i_T(\widetilde{x}) = 0 \ . \tag{21}$$

Now if $\widetilde{x}$ has a lower period, T/k say, we would have, by Corollary 31:

$$i_T(\widetilde{x}) = i_{kT/k}(\widetilde{x}) \geq k i_{T/k}(\widetilde{x}) + k - 1 \geq k - 1 \geq 1 \ . \tag{22}$$

This would contradict (21), and thus cannot happen. □

To understand the nontriviality conditions, such as the one in formula (16), one may think of a one-parameter family x_T, $T \in (2\pi\omega^{-1}, 2\pi b_\infty^{-1})$ of periodic solutions, $x_T(0) = x_T(T)$, with x_T going away to infinity when $T \to 2\pi\omega^{-1}$, which is the period of the linearized system at 0.

Year	World population
8000 B.C.	5,000,000
50 A.D.	200,000,000
1650 A.D.	500,000,000
1945 A.D.	2,300,000,000
1980 A.D.	4,400,000,000

Table 1. This is the example table taken out of *The TEXbook,* p. 246. Its caption is placed below the table.

Theorem 5 (Ghoussoub-Preiss). *Assume $H(t,x)$ is $(0,\varepsilon)$-subquadratic at infinity for all $\varepsilon > 0$, and T-periodic in t*

$$H(t,\cdot) \qquad \text{is convex} \quad \forall t \tag{23}$$

$$H(\cdot,x) \qquad \text{is} \quad T\text{−periodic} \quad \forall x \tag{24}$$

$$H(t,x) \ge n\left(\|x\|\right) \qquad \text{with} \quad n(s)s^{-1} \to \infty \quad \text{as} \quad s \to \infty \tag{25}$$

$$\forall \varepsilon > 0 \ , \quad \exists c \ : \ H(t,x) \le \frac{\varepsilon}{2} \|x\|^2 + c \ . \tag{26}$$

Assume also that H is C^2, and $H''(t,x)$ is positive definite everywhere. Then there is a sequence x_k, $k \in \mathbb{N}$, of kT-periodic solutions of the system

$$\dot{x} = JH'(t,x) \tag{27}$$

such that, for every $k \in \mathbb{N}$, there is some $p_o \in \mathbb{N}$ with:

$$p \ge p_o \Rightarrow x_{pk} \ne x_k \ . \tag{28}$$

*Example 6 (*External forcing*).* Consider the system:

$$\dot{x} = JH'(x) + f(t) \tag{29}$$

where the Hamiltonian H is $(0, b_\infty)$-subquadratic, and the forcing term is a distribution on the circle:

$$f = \frac{d}{dt}F + f_o \qquad \text{with} \quad F \in L^2\left(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n}\right) \ , \tag{30}$$

where $f_o := T^{-1}\int_o^T f(t)dt$. For instance,

$$f(t) = \sum_{k \in \mathbb{N}} \delta_k \xi \ , \tag{31}$$

where δ_k is the Dirac mass at $t = k$ and $\xi \in \mathbb{R}^{2n}$ is a constant, fits the prescription. This means that the system $\dot{x} = JH'(x)$ is being excited by a series of identical shocks at interval T.

Definition 7. *Let $A_\infty(t)$ and $B_\infty(t)$ be symmetric operators in $\mathbb{R}^{2n}$, depending continuously on $t \in [0,T]$, such that $A_\infty(t) \le B_\infty(t)$ for all t.*

A Borelian function $H : [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ is called (A_∞, B_∞)-subquadratic at infinity if there exists a function $N(t,x)$ such that:

$$H(t,x) = \frac{1}{2}\left(A_\infty(t)x, x\right) + N(t,x) \tag{32}$$

$$\forall t\ , \quad N(t,x) \quad \text{is convex with respect to} \quad x \tag{33}$$

$$N(t,x) \geq n\left(\|x\|\right) \quad \text{with} \quad n(s)s^{-1} \to +\infty \quad \text{as} \quad s \to +\infty \tag{34}$$

$$\exists c \in \mathbb{R} \ : \quad H(t,x) \leq \frac{1}{2}\left(B_\infty(t)x, x\right) + c \quad \forall x\ . \tag{35}$$

If $A_\infty(t) = a_\infty I$ and $B_\infty(t) = b_\infty I$, with $a_\infty \leq b_\infty \in \mathbb{R}$, we shall say that H is (a_∞, b_∞)-subquadratic at infinity. As an example, the function $\|x\|^\alpha$, with $1 \leq \alpha < 2$, is $(0, \varepsilon)$-subquadratic at infinity for every $\varepsilon > 0$. Similarly, the Hamiltonian

$$H(t,x) = \frac{1}{2}k\left\|k\right\|^2 + \|x\|^\alpha \tag{36}$$

is $(k, k+\varepsilon)$-subquadratic for every $\varepsilon > 0$. Note that, if $k < 0$, it is not convex.

4 Conclusions

...and wherever you travel we wish you a nice trip.

References

1. B. Baggins: *Introduction to the theory, languages, and routing*, Mulison-Wesley, Writing, Massachusetts, 1979.

2. B. Baggins: *Introduction to the theory of bags*, Mulison-Wesley, Writing, Massachusetts, 1990.