The monoid of queue actions

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Abstract. We model the behavior of a fifo-queue as a monoid of transformations that are induced by sequences of writing and reading. We describe this monoid by means of a confluent and terminating semi-Thue system and study some of its basic algebraic properties such as conjugacy. Moreover, we show that while several properties concerning its rational subsets are undecidable, their uniform membership problem is NL-complete. Furthermore, we present an algebraic characterization of this monoid's recognizable subsets. Finally, we prove that it is not Thurston-automatic.

1 Introduction

Basic computing models differ in their storage mechanisms: there are finite memory mechanisms, counters, blind counters, partially blind counters, pushdowns, Turing tapes, queues and combinations of these mechanisms. Every storage mechanism naturally comes with a set of basic actions like reading a symbol from or writing a symbol to the pushdown. As a result, sequences of basic actions transform the storage. The set of transformations induced by sequences of basic actions then forms a monoid. As a consequence, fundamental properties of a storage mechanism are mirrored by algebraic properties of the induced monoid. For example, the monoid induced by a deterministic finite automaton is finite, a single blind counter induces the integers with addition, and pushdowns induce polycyclic monoids [Kam09]. In this paper, we are interested in a queue as a storage mechanism. In particular, we investigate the monoid $\mathcal{Q}$ induced by a single queue.

The basic actions on a queue are writing the symbol a into the queue and reading the symbol a from the queue (for each symbol a from the alphabet of the queue). Since a can only be read from a queue if it is the first entry in the queue, these actions are partial. Hence, for every sequence of basic actions, there is a queue of shortest length that can be transformed by the sequence

* The results of this paper were obtained when Martin Huschenbett was affiliated with Institut für Theoretische Informatik, TU Ilmenau.

** The third author is supported by a fellowship within the Postdoc-Program of the German Academic Exchange Service (DAAD). While the results of this paper were obtained, he was affiliated with Fachbereich Informatik, Technische Universität Kaiserslautern.

without error (i.e., without attempting to read a from a queue that does not start with a). Our first main result (Theorem 4.3) in section 4 provides us with a normal form for transformations induced by sequences of basic actions: The transformation induced by a sequence of basic actions is uniquely determined by the subsequence of write actions, the subsequence of read actions, and the length of the shortest queue that can be transformed by the sequence without error. The proof is based on a convergent finite semi-Thue system for the monoid $\mathcal{Q}$. In sections 3 and 5, we derive equations that hold in $\mathcal{Q}$. The main result in this direction is Theorem 5.4, which describes the normal form of the product of two sequences of basic actions in normal form, i.e., it describes the monoid operation in terms of normal forms.

Sections 6 and 7 concentrate on the conjugacy problem in $\mathcal{Q}$. The fundamental notion of conjugacy in groups has been extended to monoids in two different ways: call x and y conjugate if the equation $xz = zy$ has a solution, and call them transposed if there are u and v such that $x = uv$ and $y = vu$. Then conjugacy $\approx$ is reflexive and transitive, but not necessarily symmetric, and transposition $\sim$ is reflexive and symmetric, but not necessarily transitive. These two relations have been considered, e.g., in [LS69,Osi73,Ott84,Dub86,Zha91,Cho93]. We prove that conjugacy is the transitive closure of transposition and that two elements of $\mathcal{Q}$ are conjugate if and only if their subsequences of write and of read actions, respectively, are conjugate in the free monoid. This characterization allows in particular to decide conjugacy in polynomial time. In section 7, we prove that the set of solutions $z \in \mathcal{Q}$ of $xz = zy$ is effectively rational but not necessarily recognizable.

Section 8 investigates algorithmic properties of rational subsets of $\mathcal{Q}$. Algorithmic aspects of rational subsets have received increased attention in recent years; see [Loh13] for a survey on the membership problem. Employing the fact that every element of $\mathcal{Q}$ has only polynomially many left factors, we can nondeterministically solve the rational subset membership problem in logarithmic space. Since the direct product of two free monoids embeds into $\mathcal{Q}$, all the negative results on rational transductions (cf. [Ber79]) as, e.g., the undecidability of universality of a rational subset, translate into our setting (cf. Theorem 8.4). The subsequent section 9 characterizes the recognizable subsets of $\mathcal{Q}$. Recall that an element of $\mathcal{Q}$ is completely determined by its subsequences of write and read actions, respectively, and the length of the shortest queue that can be transformed without an error. Regular conditions on the subsequences of write and read actions, respectively, lead to recognizable sets in $\mathcal{Q}$. Regarding the shortest queue that can be transformed without error, the situation is more complicated: the set of elements of $\mathcal{Q}$ that operate error-free on the empty queue is not recognizable. Using an approximation of the length of the shortest queue, we obtain recognizable subsets $\Omega_k \subseteq \mathcal{Q}$. The announced characterization then states that a subset of $\mathcal{Q}$ is recognizable if and only if it is a Boolean combination of regular conditions on the subsequences of write and read actions, respectively, and sets Ω_k (cf. Theorem 9.4). In the final section 10, we prove that $\mathcal{Q}$ is not automatic in the sense of Thurston et al. [CEH$^+$92] (it cannot be automatic in the sense of Khoussainov and Nerode [KN95] since the free monoid with two generators is interpretable in first order logic in $\mathcal{Q}$).

2 Preliminaries

Let A be an alphabet. As usual, the set of finite words over A, i.e. the free monoid generated by A, is denoted A^*. Let $w = a_1 \ldots a_n \in A^*$ be some word. The *length* of w is $|w| = n$. The word obtained from w by reversing the order of its symbols is $w^R = a_n \ldots a_1$. A word $u \in A^*$ is a *prefix* of w if there is $v \in A^*$ such that $w = uv$. In this situation, the word v is unique and we refer to it by $u^{-1}w$. Similarly, u is a *suffix* of w if $w = vu$ for some $v \in A^*$ and we then put $wu^{-1} = v$. For $k \in \mathbb{N}$, we let $A^{\leq k} = \{\ w \in A^* \mid |w| \leq k\,\}$ and define $A^{>k}$ similarly.

Let M be an arbitrary monoid. The *concatenation* of two subsets $X, Y \subseteq M$ is defined as $X \cdot Y = \{\, xy \mid x \in X, y \in Y \,\}$. The *Kleene iteration* of X is the set $X^* = \{\, x_1 \cdots x_n \mid n \in \mathbb{N}, x_1, \ldots, x_n \in X \,\}$. In fact, X^* is a submonoid of M, namely the smallest submonoid entirely including X. Thus, X^* is also called *the submonoid generated by* X. The monoid M is *finitely generated*, if there is some finite subset $X \subseteq M$ such that $M = X^*$.

A subset $L \subseteq M$ is called *rational* if it can be constructed from the finite subsets of M using union, concatenation, and Kleene iteration only. The subset L is *recognizable* if there are a finite monoid F and a morphism $\phi\colon M \to F$ such that $\phi^{-1}(\phi(L)) = L$. The image of a rational set under a monoid morphism is again rational, whereas recognizability is retained under preimages of morphisms. It is well-known that every recognizable subset of a finitely generated monoid is rational. The converse implication is in general false. However, if M is the free monoid generated by some alphabet A, a subset $L \subseteq A^*$ is rational if and only if it is recognizable. In this situation, we call L *regular*.

3 Definition and basic equations

We want to model the behavior of a fifo-queue whose entries come from a finite set A with $|A| \geq 2$ (if A is a singleton, the queue degenerates into a partially blind counter). Consequently, the state of a valid queue is an element from A^*. In order to have a defined result even if a read action fails, we add the error state $\bot$. The basic actions are writing of the symbol $a \in A$ into the queue (denoted a) and reading the symbol $a \in A$ from the queue (denoted $\overline{a}$). Formally, $\overline{A}$ is a disjoint copy of A whose elements are denoted $\overline{a}$. Furthermore, we set $\Sigma = A \cup \overline{A}$. Hence, the free monoid Σ^* is the set of sequences of basic actions and it acts on the set $A^* \cup \{\bot\}$ by way of the function $.\colon (A^* \cup \{\bot\}) \times \Sigma^* \to A^* \cup \{\bot\}$, which is defined as follows:

$$q.\varepsilon = q \qquad q.au = qa.u \qquad q.\overline{a}u = \begin{cases} q'.u & \text{if } q = aq' \\ \bot & \text{otherwise} \end{cases} \qquad \bot.u = \bot$$

for $q \in A^*$, $a \in A$, and $u \in \Sigma^*$.

Example 3.1. Let the content of the queue be $q = ab$. Then $ab.\overline{a}c = b.c = bc.\varepsilon = bc$ and $ab.c\overline{a} = abc.\overline{a} = bc.\varepsilon = bc$, i.e., the sequences of basic actions $\overline{a}c$ and $c\overline{a}$ behave the same on the queue $q = ab$. In Lemma 3.5, we will see that this is the case for any queue $q \in A^* \cup \{\bot\}$. Differently, we have $\varepsilon.\overline{a}a = \bot \neq \varepsilon = \varepsilon.a\overline{a}$, i.e., the sequences of basic actions $a\overline{a}$ and $\overline{a}a$ behave differently on certain queues.

Definition 3.2. *Two words $u, v \in \Sigma^*$ are* equivalent *if $q.u = q.v$ for all queues $q \in A^*$. In that case, we write $u \equiv v$. The equivalence class wrt. $\equiv$ containing the word u is denoted $[u]$.*

Since $\equiv$ is a congruence on the free monoid Σ^*, we can define the quotient monoid $\mathcal{Q} = \Sigma^*/\equiv$ and the natural epimorphism $\eta\colon \Sigma^* \to \mathcal{Q}\colon w \mapsto [w]$. The monoid $\mathcal{Q}$ is called the monoid of queue actions.

Intuitively, the basic actions a and $\overline{a}$ act "dually" on $A^* \cup \{\bot\}$. We formalize this intuition by means of the *duality map* $\delta\colon \Sigma^* \to \Sigma^*$, which is defined as follows: $\delta(\varepsilon) = \varepsilon$, $\delta(au) = \delta(u)\overline{a}$, and $\delta(\overline{a}u) = \delta(u)a$ for $a \in A$ and $u \in \Sigma^*$. Notice that $\delta(uv) = \delta(v)\delta(u)$ and $\delta(\delta(u)) = u$ (i.e., δ is a bijective antimorphism and an involution). The following lemma captures the fundamental idea behind the principle of duality:

Lemma 3.3. *Let $p, q \in A^*$ and $u \in \Sigma^*$. Then $p.u = q$ implies $q^R.\delta(u) = p^R$.*

Proof. For $u = \varepsilon$, the claim is trivial. To proceed by induction on the length of u, let $u = cv$ with $c \in \Sigma$.

First, suppose $c = a \in A$. Then $q = p.cv = pa.v$ and hence, by the induction hypothesis, $q^R.\delta(v) = ap^R$. Consequently, $q^R.\delta(u) = q^R.\delta(v)\overline{a} = ap^R.\overline{a} = p^R$.

Next, suppose $c = \overline{a} \in \overline{A}$. Then $p.\overline{a}v = q \neq \bot$ implies the existence of some $r \in A^*$ with $p = ar$. Hence $q = p.\overline{a}v = r.v$. By the induction hypothesis, this yields $q^R.\delta(v) = r^R$ and therefore $q^R.\delta(u) = q^R.\delta(v)a = r^R.a = r^Ra = p^R$. $\square$

In the following, we use the term "by duality" to refer to the proposition below.

Proposition 3.4. *For $u, v \in \Sigma^*$, we have $u \equiv v$ if and only if $\delta(u) \equiv \delta(v)$.*

Proof. First, suppose that $u \equiv v$ and consider some arbitrary $q \in A^*$. If $q.\delta(u) = p \neq \bot$, then $p^R.u = p^R.\delta(\delta(u)) = q^R$ because of $\delta(\delta(u)) = u$ and Lemma 3.3. From $u \equiv v$, we infer $p^R.v = q^R$. Again by Lemma 3.3, we obtain $q.\delta(v) = p = q.\delta(u)$. By symmetry, $q.\delta(v) \neq \bot$ implies $q.\delta(u) = q.\delta(v)$. Hence, $q.\delta(u) = q.\delta(v)$ for all $q \in A^* \cup \{\bot\}$, i.e., $\delta(u) \equiv \delta(v)$.

Since δ is an involution, we also get the converse implication. $\square$

Proposition 3.4 permits us to lift the duality map δ from Σ^* to $\mathcal{Q}$: Since $u \equiv v$ implies $\delta(u) \equiv \delta(v)$, the lifted map $\delta\colon \mathcal{Q} \to \mathcal{Q}$ with $\delta([w]) = [\delta(w)]$ is well-defined. Observe that since δ is an involution on Σ^*, it is also an involution on $\mathcal{Q}$ satisfying $\delta(xy) = \delta(y)\delta(x)$ for all $x, y \in \mathcal{Q}$.

In the following lemma, Eq. (2) follows immediately from Eq. (1) by duality.

Lemma 3.5. *Let $a, b \in A$. Then we have*

$$a b \overline{b} \equiv a \overline{b} b \tag{1}$$
$$a \overline{a} \overline{b} \equiv \overline{a} a \overline{b} \tag{2}$$
$$a \overline{b} \equiv \overline{b} a \text{ if } a \neq b \,. \tag{3}$$

From Eq. (1) and Eq. (3), we get $ab\overline{c} \equiv a\overline{c}b$ for any $a, b, c \in A$, even when $b = c$. Similarly, Eq. (2) and Eq. (3) imply $a\overline{b}\overline{c} \equiv \overline{b}a\overline{c}$.

Proof. To prove $q.ab\overline{b} = q.a\overline{b}b$ (i.e., Eq. (1)), we distinguish the cases $qa \in bA^*$ and $qa \notin bA^*$ (note that $a = b$ is not excluded). Suppose $qa = bq' \in bA^*$, then $q.ab\overline{b} = qab.\overline{b} = q'b$ and $q.a\overline{b}b = bq'.\overline{b}b = q'b$. Next let $qa \notin bA^*$ such that $qab \notin bA^*$. Then $q.ab\overline{b} = qab.\overline{b} = \perp$ and $q.a\overline{b}b = (qa.\overline{b}).b = \perp$. This proves Eq. (1).

Due to Eq. (1), we have $ba\overline{a} \equiv b\overline{a}a$. According to Proposition 3.4, this implies $a\overline{a}\overline{b} = \delta(ba\overline{a}) \equiv \delta(b\overline{a}a) = \overline{a}a\overline{b}$, i.e., Eq. (2).

Finally, suppose $a \neq b$. If $q = bq' \in bA^*$, then $q.a\overline{b} = qa.\overline{b} = q'a = q.\overline{b}a$. Next consider the case $q \notin bA^*$. Then $q.a\overline{b} = qa.\overline{b} = \perp$ since $qa \notin bA^*$ (the case $q = \varepsilon$ uses $a \neq b$). Similarly $q.\overline{b}a = \perp$ since $q \notin bA^*$. Hence $a\overline{b} \equiv \overline{b}a$, i.e., Eq. (3) holds. ☐

Our computations in $\mathcal{Q}$ will frequently make use of alternating sequences of write- and read-operations on the queue. To simplify notation, we define the shuffle of two words over A and over $\overline{A}$ as follows: Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in A$ with $v = a_1 a_2 \ldots a_n$ and $w = b_1 b_2 \ldots b_n$. We write $\overline{w}$ for $\overline{b_1}\,\overline{b_2}\,\ldots\,\overline{b_n}$ and set

$$\langle v, \overline{w} \rangle = a_1 \overline{b_1}\, a_2 \overline{b_2}\, \ldots\, a_n \overline{b_n}$$

(note that $\langle v, \overline{w} \rangle$ is only defined if v and w are words over A of equal length).

Lemma 3.6. *Let $u, v \in A^*$ and $a, b \in A$.*

(1) If $|u| = |av|$, then $\langle u, \overline{av} \rangle\, \overline{b} \equiv \overline{a}\, \langle u, \overline{vb} \rangle$.
(2) If $|ub| = |v|$, then $a\, \langle ub, \overline{v} \rangle \equiv \langle au, \overline{v} \rangle\, b$.
(3) If $|u| = |v|$, then $a\, \langle u, \overline{v} \rangle\, \overline{b} \equiv \langle au, \overline{vb} \rangle$.

We remark that the equations in (1) and (2) are dual and that the equation in (3) is self-dual.

Proof. We prove the first claim by induction on the length of v (which equals $|u| - 1$): if $|v| = 0$, then $u \in A$ and therefore $\langle u, \overline{av} \rangle\, \overline{b} = u\overline{a}\overline{b} \equiv \overline{a}u\overline{b} = \overline{a}\, \langle u, \overline{vb} \rangle$ by Lemma 3.5(2). Next let $|v| > 0$. Then there exist $v_1, u_1 \in A$ and $v_2, u_2 \in A^*$ with $v = v_1 v_2$ and $u = u_1 u_2$. We obtain

$$\begin{aligned}
\langle u, \overline{av} \rangle\, \overline{b} &= u_1 \overline{a}\, \langle u_2, \overline{v_1 v_2} \rangle\, \overline{b} \\
&\equiv u_1 \overline{a}\, \overline{v_1}\, \langle u_2, \overline{v_2 b} \rangle && \text{(by the induction hypothesis)} \\
&\equiv \overline{a} u_1\, \overline{v_1}\, \langle u_2, \overline{v_2 b} \rangle && \text{(by Lemma 3.5(2))} \\
&= \overline{a}\, \langle u, \overline{vb} \rangle \;.
\end{aligned}$$

This finishes the proof of the first claim, the second follows by duality: Let $|ub| = |v|$. Then

$$\begin{aligned}
a\,\langle ub, \overline{v}\rangle &= \delta\left(\left\langle v^R, \overline{bu^R}\right\rangle \overline{a}\right) \\
&\equiv \delta\left(\overline{b}\left\langle v^R, \overline{u^R a}\right\rangle\right) && \text{(by the first claim and Prop. 3.4)} \\
&= \langle au, \overline{v}\rangle\, b\,.
\end{aligned}$$

The third statement is trivial for $|v| = 0$. If $|v| > 0$, there are $v_1 \in A$ and $v_2 \in A^*$ with $v = v_1 v_2$. Then we get from the first statement

$$a\,\langle u, \overline{v}\rangle\,\overline{b} \equiv a\overline{v_1}\,\left\langle u, \overline{v_2 b}\right\rangle = \left\langle au, \overline{vb}\right\rangle\,. \qquad \square$$

By induction on the length of y, one obtains the following generalizations (for (2), induction on the length of x is used).

Proposition 3.7. *Let $u, v, x, y, x', y' \in A^*$.*

(1) if $xy = x'y'$ and $|x| = |y'| = |u|$, then $\langle u, \overline{x}\rangle\,\overline{y} \equiv \overline{x'}\,\langle u, \overline{y'}\rangle$.
(2) if $xy = x'y'$ and $|y| = |x'| = |v|$, then $x\,\langle y, \overline{v}\rangle \equiv \langle x', \overline{v}\rangle\, y'$.
(3) If $|u| = |v|$ and $|x| = |y|$, then $x\,\langle u, \overline{v}\rangle\,\overline{y} \equiv \langle xu, \overline{vy}\rangle$.
(4) If $|x| = |y|$, then $\langle x, \overline{y}\rangle \equiv x\overline{y}$.

We note that, again, the equations in (1) and in (2) are dual and the ones in (3) and (4) are self-dual. Moreover, (4) is a special case of (3) for $u = v = \varepsilon$.

Corollary 3.8. *Let $u, v, w \in A^*$.*

(1) If $|w| = |v|$, then $\overline{u}v\overline{w} \equiv v\overline{uw}$.
(2) If $|u| = |v|$, then $u\overline{v}w \equiv uw\overline{v}$.

In this corollary, the second statement is the dual of the first.

Proof. We prove the first claim. Let $u = b_1 b_2 \dots b_m$ and $w = b_{m+1} b_{m+2} \dots b_{m+n}$ with $b_i \in A$ for all $1 \le i \le m+n$. Note that $n = |w| = |v|$. Then we have

$$\begin{aligned}
\overline{u}v\overline{w} &\equiv \overline{b_1 \dots b_m}\,\left\langle v, \overline{b_{m+1} \dots b_{m+|v|}}\right\rangle && \text{(by Prop. 3.7 (4))} \\
&\equiv \left\langle v, \overline{b_1 \dots b_{|v|}}\right\rangle \overline{b_{|v|+1} \dots b_{m+n}} && \text{(by Prop. 3.7 (1))} \\
&\equiv v\overline{b_1 \dots b_{|v|}}\,\overline{b_{|v|+1} \dots b_{m+n}} && \text{(by Prop. 3.7 (3))} \\
&= v\overline{uw}\,.
\end{aligned}$$

The second statement follows by duality: Let $|u| = |v|$. Then

$$\begin{aligned}
u\overline{v}w &= \delta(\delta(w)\delta(\overline{v})\delta(u)) \\
&\equiv \delta(\delta(\overline{v})\delta(w)\delta(u)) && \text{(by the first claim and Prop. 3.4)} \\
&= uw\overline{v}\,.
\end{aligned}$$

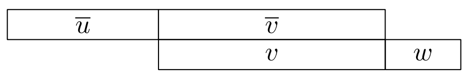
$\square$

4 A semi-Thue system for the queue monoid $\mathcal{Q}$

We order the equations from Lemma 3.5 as follows:

$$
\begin{aligned}
a\overline{b} &\to \overline{b}a \text{ for } a \neq b \\
ab\overline{b} &\to a\overline{b}b \\
a\overline{a}\overline{b} &\to \overline{a}a\overline{b}
\end{aligned}
$$

Let R be the semi-Thue system with the above three types of rules. Note that a word over Σ is irreducible if and only if it has the form $\overline{u}\ \langle v, \overline{v}\rangle\ w$ for some $u, v, w \in A^*$. When doing our calculations, we found it convenient to think about the irreducible word $\overline{u}\,\langle v, \overline{v}\rangle\, w$ in terms of pictures as follows:



Here, the blocks represent the words $\overline{u}$, $\overline{v}$, v, and w, respectively. We placed the read-blocks (i.e., words over $\overline{A}$) in the first line and write-blocks in the second. The shuffle $\langle v, \overline{v}\rangle$ is illustrated by placing the corresponding two blocks on top of each other.

Lemma 4.1. *The semi-Thue system R is terminating and confluent.*

Proof. We first show termination: For this, order the alphabet Σ such that $\overline{a} < b$ for all $a, b \in A$. Then, for any rule $u \to v$ from R, the word v is length-lexicographically properly smaller than u. Since the set Σ^* ordered length-lexicographically is isomorphic to $(\mathbb{N}, \leq)$, the semi-Thue system R is terminating.

To prove confluence of R, it suffices to show that R is locally confluent. Note that the only overlap of two left-hand sides of R has the form $ab\overline{b}\overline{c}$ with $a, b, c \in A$. In this case, we can apply two rules (namely $ab\overline{b} \to a\overline{b}b$ and $b\overline{b}\overline{c} \to \overline{b}b\overline{c}$) which, in both cases, results in $a\overline{b}b\overline{c}$. $\square$

Let $u \in \Sigma^*$. Since R is terminating and confluent, there is a unique irreducible word $\mathsf{nf}(u)$ with $u \xrightarrow{*} \mathsf{nf}(u)$. We call $\mathsf{nf}(u)$ the *normal form* of u and denote the set of all normal forms by $\mathsf{NF} \subseteq \Sigma^*$, i.e.,

$$\mathsf{NF} = \{\ \mathsf{nf}(u) \mid u \in \Sigma^*\ \} = \overline{A}^*\ \{\, a\overline{a} \mid a \in A\,\}^*\ A^*\,.$$

Note that, by Lemma 3.5, we have $u \equiv \mathsf{nf}(u)$. Consequently, $\mathsf{nf}(u) = \mathsf{nf}(v)$ implies $u \equiv v$ for any words $u, v \in \Sigma^*$. We next prove the converse implication.

Lemma 4.2. *Let $u, v \in \Sigma^*$ with $u \equiv v$. Then $\mathsf{nf}(u) = \mathsf{nf}(v)$.*

Proof. Let $\mathsf{nf}(u) = \overline{u_1}\,\langle u_2, \overline{u_2}\rangle\, u_3$ and $\mathsf{nf}(v) = \overline{v_1}\,\langle v_2, \overline{v_2}\rangle\, v_3$ and recall that $u \equiv \mathsf{nf}(u) \equiv \overline{u_1}u_2\overline{u_2}u_3$ holds by Prop. 3.7(4). Hence, in the following, we may assume $u = \overline{u_1}u_2\overline{u_2}u_3$ and similarly $v = \overline{v_1}v_2\overline{v_2}v_3$.

We first show $u_1 = v_1$ by contradiction. So suppose $u_1 \neq v_1$ and, without loss of generality, $|u_1| \leq |v_1|$. Then consider $q = u_1$. We get $q.u = \varepsilon.u_2\overline{u_2}\,u_3 = u_3$. Furthermore, $u_1.\overline{v_1} = \perp$ since $u_1 \neq v_1$ and $|u_1| \leq |v_1|$. Consequently $q.v = (q.v_1).v_2\overline{v_2}\,v_3 = \perp$. This contradicts the assumption $q.u = q.v$, so we obtain $u_1 = v_1$.

Without loss of generality, we may assume $|u_2| \leq |v_2|$. Then we get

$$\begin{aligned} \perp \neq u_2u_3 &= u_1u_2.\overline{u_1}u_2\overline{u_2}u_3 \\ &= u_1u_2.\overline{v_1}v_2\overline{v_2}v_3 && \text{(since } u \equiv v\text{)} \\ &= u_2.v_2\overline{v_2}v_3 && \text{(since } u_1 = v_1\text{)} \\ &= (u_2.v_2\overline{v_2}).v_3\,. \end{aligned}$$

Hence $\perp \neq u_2.v_2\overline{v_2} = u_2v_2.\overline{v_2}$. It follows that v_2 is a prefix of u_2v_2 and, since $|u_2| \leq |v_2|$, the word u_2 is a prefix of v_2. By contradiction, suppose u_2 is a proper prefix of v_2. Since $|A| \geq 2$, there exists $a \in A$ such that u_2a is no prefix of v_2 (but still $|u_2a| \leq |v_2|$). Then we get

$$u_1u_2a.u = u_1u_2a.\overline{u_1}u_2\overline{u_2}u_3 = u_2a.u_2\overline{u_2}u_3 = au_2u_3 \neq \perp$$

and

$$u_1u_2a.v = u_1u_2a.\overline{v_1}v_2\overline{v_2}v_3 = u_2a.v_2\overline{v_2}v_3 = u_2av_2.\overline{v_2}v_3 = \perp$$

which contradicts the assumption $u \equiv v$. Hence $u_2 = v_2$.

To finally show $u_3 = v_3$, consider the queue $q = u_1$. Then

$$u_3 = \varepsilon.u_2\overline{u_2}u_3 = u_1.\overline{u_1}u_2\overline{u_2}u_3 = u_1.\overline{v_1}v_2\overline{v_2}v_3 = \varepsilon.v_2\overline{v_2}v_3 = v_3\,. \qquad \square$$

The above two lemmas ensure that $u \equiv v$ and $\mathsf{nf}(u) = \mathsf{nf}(v)$ are equivalent. Hence, the mapping $\mathsf{nf}\colon \Sigma^* \to \mathsf{NF}$ can be lifted to a mapping $\mathsf{nf}\colon \mathcal{Q} \to \mathsf{NF}$ by defining $\mathsf{nf}([u]) = \mathsf{nf}(u)$.

Theorem 4.3. *The natural epimorphism $\eta\colon \Sigma^* \to \mathcal{Q}$ maps the set NF bijectively onto $\mathcal{Q}$. The inverse of this bijection is the map $\mathsf{nf}\colon \mathcal{Q} \to \mathsf{NF}$.*

This theorem allows us to define projection maps on $\mathcal{Q}$. First, the morphisms $\pi, \overline{\pi}\colon \Sigma^* \to A^*$ are defined by $\pi(a) = \overline{\pi}(\overline{a}) = a$ and $\pi(\overline{a}) = \overline{\pi}(a) = \varepsilon$ for $a \in A$. In other words, π is the projection of a word over Σ to its subword over A, and $\overline{\pi}$ is the projection to its subword over $\overline{A}$, with all the bars $^-$ deleted. E.g., $\pi(a\overline{b}\overline{a}b) = ab$ and $\overline{\pi}(a\overline{b}\overline{a}b) = ba$. From Theorem 4.3, we learn that $u \equiv v$ implies $\pi(u) = \pi(v)$ and $\overline{\pi}(u) = \overline{\pi}(v)$. Hence, π and $\overline{\pi}$ can be lifted to morphisms $\pi, \overline{\pi}\colon \mathcal{Q} \to A^*$ by $\pi([u]) = \pi(u)$ and $\overline{\pi}([u]) = \overline{\pi}(u)$.

Notice that the two projections $\pi(q)$ and $\overline{\pi}(q)$ of a queue action $q \in \mathcal{Q}$ do not entirely determine q, e.g., $[\overline{a}a] \neq [a\overline{a}]$. However they clearly do in combination with the "overlap width of q":

Definition 4.4. *Let $w \in \Sigma^*$ be a word and $\mathsf{nf}(w) = \overline{x}\,\langle y, \overline{y}\rangle\, z$ its normal form. The* overlap width *of w and of $[w]$ is the number*

$$\mathsf{ow}(w) = \mathsf{ow}([w]) = |y|\,.$$

Observation 4.5 *Every $q \in \mathcal{Q}$ is completely described by $\pi(q)$, $\overline{\pi}(q)$, and $\mathsf{ow}(q)$.*

Remark. Let $q \in \mathcal{Q}$ and $w = \mathsf{nf}(q) = \overline{x}\,\langle y, \overline{y}\rangle\, z$ its normal form. Then $x.\overline{x}\,\langle y, \overline{y}\rangle\, z = \varepsilon.\langle y, \overline{y}\rangle\, z = \varepsilon.z = z$, i.e., q transforms the queue x without error. On the other hand, if w acts on a queue x' without error, then x is a prefix of x'. Hence $|x|$ is the length of the shortest queue that can be transformed by q without error. Since $\mathsf{ow}(q) = |\pi(q)| - |x|$, q is also uniquely given by $\pi(q)$, $\overline{\pi}(p)$, and the length of the shortest queue which is transformed by q without error.

From Theorem 4.3, we can derive two other properties of $\mathcal{Q}$. A monoid M is called *left-cancellative* if $xy = xz$ implies $y = z$ for $x, y, z \in M$. It is called *right-cancellative*, if $yx = zx$ implies $y = z$ for $x, y, z \in M$.

Corollary 4.6. *The monoid $\mathcal{Q}$ is neither left- nor right-cancellative.*

Proof. Observe that $[\overline{a}a] \neq [a\overline{a}]$, because both $\overline{a}a$ and $a\overline{a}$ are in normal form. This means, the identities $[a][\overline{a}a] = [a][a\overline{a}]$ and $[a\overline{a}][\overline{a}] = [\overline{a}a][\overline{a}]$ show that $\mathcal{Q}$ is neither left- nor right-cancellative. $\square$

An important concept in the theory of semigroups is that of Green's relations, which are defined as follows. Let M be a monoid and $x, y \in M$. We write

$$\begin{aligned}
x\mathcal{L}y &\quad \text{if } Mx = My,\\
x\mathcal{R}y &\quad \text{if } xM = yM,\\
x\mathcal{H}y &\quad \text{if } x\mathcal{L}y \text{ and } x\mathcal{R}y,\\
x\mathcal{D}y &\quad \text{if } x\mathcal{L}z \text{ and } z\mathcal{R}y \text{ for some } z \in M,\\
x\mathcal{J}y &\quad \text{if } MxM = MyM.
\end{aligned}$$

We observe here that for the monoid $\mathcal{Q}$, all these relations coincide with the identity relation. The monoid M is called *$\mathcal{J}$-trivial* if $x\mathcal{J}y$ implies $x = y$.

Corollary 4.7. *$\mathcal{Q}$ is $\mathcal{J}$-trivial.*

Proof. Let $s, t \in \mathcal{Q}$ with $s\mathcal{J}t$, i.e., $\mathcal{Q}s\mathcal{Q} = \mathcal{Q}t\mathcal{Q}$. Then there are $u, u' \in \Sigma^*$ with $[u]s[u'] = t$ since $t \in \mathcal{Q}t\mathcal{Q} = \mathcal{Q}s\mathcal{Q}$. Similarly, there are $v, v' \in \Sigma^*$ with $[v]t[v'] = s$. Hence $s = [v]t[v'] = [vu]s[u'v']$. However, this implies $|uvu'v'| = 0$, i.e., $s = t$. $\square$

Since each of the other relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ implies $\mathcal{J}$, it follows that all of them are trivial as well.

5 Multiplication

For two words $u, v \in \Sigma^*$ in normal form, we want to determine the normal form of uv. For this, the concept of *overlap* of two words will be important:

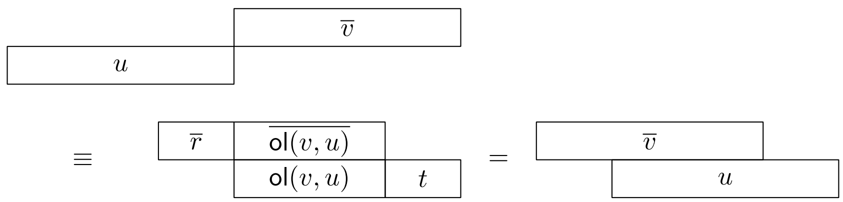
Definition 5.1. *For $u, v \in A^*$, let $\mathsf{ol}(v, u)$ denote the longest suffix of v that is also a prefix of u.*

Example. $\mathsf{ol}(ab, bc) = b$, $\mathsf{ol}(aba, aba) = aba$, and $\mathsf{ol}(ab, cba) = \varepsilon$.

Lemma 5.2. *Let $u, v \in A^*$ with $|u| = |v|$ and set $s = \mathsf{ol}(v, u)$, $r = vs^{-1}$ and $t = s^{-1}u$. Then*

$$u\overline{v} \equiv \overline{r}\,\langle s, \overline{s}\rangle\, t\,, i.e.,\ \mathsf{nf}(u\overline{v}) = \overline{r}\,\langle s, \overline{s}\rangle\, t\,.$$

The equation $u\overline{v} \equiv \overline{r}\,\langle s, \overline{s}\rangle\, t$ can be visualized as follows:



Our intuition is that the word $\overline{v}$ tries to slide along u to the left as far as possible. This movement is stopped as soon as we reach a word in normal form. This happens, for the first time, when the suffix $\mathsf{ol}(v, u)$ of v aligns with the prefix $\mathsf{ol}(v, u)$ of u.

Proof. Let $u = a_1 a_2 \dots a_n$ and $v = b_1 b_2 \dots b_n$ with $a_i, b_i \in A$ for all $1 \le i \le n$. We prove the statement by induction on n. For $n = 0$, the statement is trivial, so we may assume $n > 0$. If $u = v$, we have $\mathsf{ol}(v, u) = u$, confirming the equation by Prop. 3.7(4). If $u \neq v$, there is some $i \in \{1, 2, \dots, n\}$ such that $a_i \neq b_i$. Then we have

$$\begin{aligned}
\langle u, \overline{v}\rangle &= \left\langle a_1 \dots a_{i-1}, \overline{b_1 \dots b_{i-1}}\right\rangle a_i\overline{b_i}\left\langle a_{i+1} \dots a_n, \overline{b_{i+1} \dots b_n}\right\rangle \\
&\equiv \left\langle a_1 \dots a_{i-1}, \overline{b_1 \dots b_{i-1}}\right\rangle \overline{b_i}a_i\left\langle a_{i+1} \dots a_n, \overline{b_{i+1} \dots b_n}\right\rangle && \text{(Lemma 3.5(3))} \\
&\equiv \overline{b_1}\left\langle a_1 \dots a_{i-1}, \overline{b_2 \dots b_i}\right\rangle \left\langle a_i \dots a_{n-1}, \overline{b_{i+1} \dots b_n}\right\rangle a_n && \text{(Lemma 3.6(1))} \\
&= \overline{b_1}\left\langle a_1 \dots a_{n-1}, \overline{b_2 \dots b_n}\right\rangle a_n\,. && \text{(Lemma 3.6(2))}
\end{aligned}$$

Let $u' = a_1 \cdots a_{n-1}$ and $v' = b_2 \cdots b_n$. Then the induction hypothesis guarantees

$$\left\langle u', \overline{v'}\right\rangle \equiv \overline{r'}\left\langle s', \overline{s'}\right\rangle t' \text{ for } s' = \mathsf{ol}(v', u'),\ r' = v's'^{-1},\ t' = s'^{-1}u'\,.$$

Consequently, we have

$$\langle u, \overline{v}\rangle \equiv \overline{b_1}\,\overline{r'}\left\langle s', \overline{s'}\right\rangle t'\,a_n\,.$$

Since $u \neq v$ and $|u| = |v|$, we have $\mathsf{ol}(v, u) = \mathsf{ol}(v', u')$ and hence $s = s'$. This means $b_1 r' = b_1 v' s^{-1} = vs^{-1} = r$ and $t' a_n = s^{-1}u' a_n = s^{-1}u = t$. Thus,

$$\langle u, \overline{v}\rangle \equiv \overline{b_1 r'}\,\langle s, \overline{s}\rangle\, t' a_n = \overline{r}\,\langle s, \overline{s}\rangle\, t\,. \qquad \square$$

•

We next show that the above lemma holds even without the assumption $|u| = |v|$.

Lemma 5.3. *Let $u, v \in A^*$ and set $s = \mathsf{ol}(v, u)$, $r = vs^{-1}$ and $t = s^{-1}u$. Then*

$$u\overline{v} \equiv \overline{r}\,\langle s, \overline{s}\rangle\, t\,.$$

Proof. First, we assume $|u| \leq |v|$ and write $v = xy$ with $|y| = |u|$. Then Corollary 3.8 yields $u\overline{v} = u\overline{xy} \equiv \overline{x}u\overline{y}$ and by Lemma 5.2, we have

$$u\overline{y} \equiv \overline{r'}\,\left\langle s', \overline{s'}\right\rangle t' \text{ for } s' = \mathsf{ol}(y, u),\; r' = ys'^{-1},\; t = s'^{-1}u\,.$$

Since $|u| = |y|$, we have $s = \mathsf{ol}(xy, u) = \mathsf{ol}(y, u) = s'$. Furthermore, $xr' = xys'^{-1} = vs^{-1} = r$ and $t' = s'^{-1}u = s^{-1}u = t$. Hence

$$u\overline{v} \equiv \overline{x}u\overline{y} \equiv \overline{x}\,\overline{r'}\,\left\langle s', \overline{s'}\right\rangle t' = \overline{r}\,\langle s, \overline{s}\rangle\, t$$

is the desired equality.

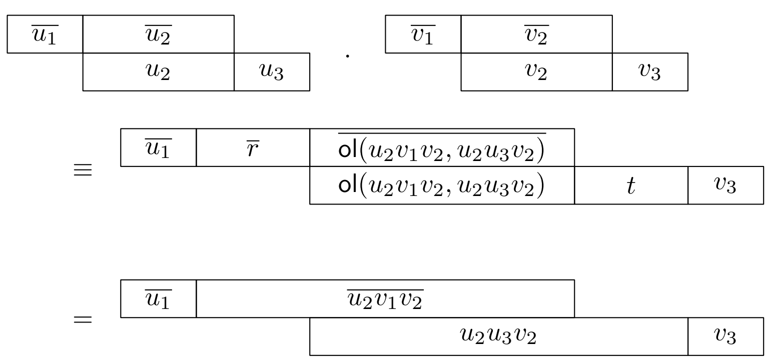
The case $|u| > |v|$ is handled by duality: define $s = \mathsf{ol}(u^R, v^R)$, $r = u^R s^{-1}$, and $t = s^{-1}v^R$. Then, by what we showed above, $u\overline{v} = \delta(v^R\overline{u^R}) = \delta(\overline{r}\,\langle s, \overline{s}\rangle\, t) = \overline{t^R}\left\langle s^R, \overline{s^R}\right\rangle r^R$. Note that $s^R = \mathsf{ol}(v, u)$, $r^R = s^{R^{-1}}u$, and $t^R = vs^{R^{-1}}$. $\square$

Finally, we describe the normal form of the product of two words in normal form. In other words, we describe the multiplication of $\mathcal{Q}$ in terms of words in normal form.

Theorem 5.4. *Let $u_1, u_2, u_3, v_1, v_2, v_3 \in A^*$ and set $s = \mathsf{ol}(u_2v_1v_2, u_2u_3v_2)$, $r = u_2v_1v_2s^{-1}$, and $t = s^{-1}u_2u_3v_2$. Then*

$$\overline{u_1}\,\langle u_2, \overline{u_2}\rangle\, u_3 \cdot \overline{v_1}\,\langle v_2, \overline{v_2}\rangle\, v_3 \equiv \overline{u_1 r}\,\langle s, \overline{s}\rangle\, tv_3\,.$$

This theorem can be visualized as follows:



Proof. We have

$$\begin{aligned}
\overline{u_1}\,\langle u_2, \overline{u_2}\rangle\, u_3 \cdot \overline{v_1}\,\langle v_2, \overline{v_2}\rangle\, v_3 &\equiv \overline{u_1}u_2\overline{u_2}u_3\overline{v_1}v_2\overline{v_2}v_3 && \text{(Prop. 3.7(4))}\\
&\equiv \overline{u_1}u_2u_3\overline{u_2}\,\overline{v_1}v_2\overline{v_2}v_3 && \text{(Cor. 3.8(2))}\\
&\equiv \overline{u_1}u_2u_3v_2\overline{u_2}\,\overline{v_1}\,\overline{v_2}v_3 && \text{(Cor. 3.8(1))}\\
&\equiv \overline{u_1}\,\overline{r}\,\langle s, \overline{s}\rangle\, tv_3\,. && \text{(Lemma 5.3)}
\end{aligned}$$

$\square$

As a consequence of Theorems 4.3 and 5.4, we can show that the queue-monoid with two letters contains all other queue-monoids as submonoids.

Corollary 5.5. *Let $\mathcal{Q}_n$ be the queue-monoid defined by an alphabet with n letters. Then $\mathcal{Q}_n$ embeds into $\mathcal{Q}_2$ for any $n \in \mathbb{N}$.*

Proof. Let $\mathcal{Q}_n$ be generated by the set $A = \{\alpha_i \mid 1 \leq i \leq n\}$ and let $\mathcal{Q}_2$ be generated by $B = \{a, b\}$. Then define a morphism $\phi\colon (A \cup \overline{A})^* \to (B \cup \overline{B})^*$ by $\phi(\alpha_i) = a^{n+i}ba^{n-i}b$ and $\phi(\overline{\alpha_i}) = \overline{a^{n+i}ba^{n-i}b}$. If $1 \leq i, j \leq n$ are distinct, then no non-empty suffix of $\phi(\alpha_i)$ is a prefix of $\phi(\alpha_j)$, i.e., $\mathsf{ol}(\phi(\alpha_i), \phi(\alpha_j)) = \varepsilon$. Hence $\phi(\alpha_i\overline{\alpha_j}) \equiv \phi(\overline{\alpha_j}\alpha_i)$ by Theorem 5.4. Furthermore note that all the words $\phi(\alpha_i)$ have length $2n + 2$. Consequently, by Cor. 3.8, we have $\phi(\alpha_i\alpha_j\overline{\alpha_j}) \equiv \phi(\alpha_i\overline{\alpha_j}\alpha_j)$ and $\phi(\alpha_i\overline{\alpha_i}\overline{\alpha_j}) \equiv \phi(\overline{\alpha_i}\alpha_i\overline{\alpha_j})$ for all $1 \leq i, j \leq n$. From these observations and Theorem 4.3, we get $\phi(U) \equiv \phi(U')$ for all $U, U' \in (A \cup \overline{A})^*$ with $U \equiv U'$.

For the converse direction, let $\phi'\colon A^* \cup \{\bot\} \to B^* \cup \{\bot\}$ be the function with $\phi'|_{A^*} = \phi$ and $\phi'(\bot) = \bot$. Observe that for $q \in A^* \cup \{\bot\}$, we have

$$\phi'(q).\phi(\alpha_i) = \phi'(q.\alpha_i), \qquad\qquad \phi'(q).\phi(\overline{\alpha_i}) = \phi'(q.\overline{\alpha_i}),$$

which, by induction, implies $\phi'(q).\phi(U) = \phi'(q.U)$ for every $U \in (A \cup \overline{A})^*$. Therefore, if $\phi(U) \equiv \phi(U')$, then for every $q \in A^* \cup \{\bot\}$, we have

$$\phi'(q.U) = \phi'(q).\phi(U) = \phi'(q).\phi(U') = \phi'(q.U').$$

Since ϕ' is injective, this yields $q.U = q.U'$ and thus $U \equiv U'$.

In other words, the morphism $\phi\colon A^* \to B^*$ can be lifted to an injective morphism from $\mathcal{Q}_n$ to $\mathcal{Q}_2$, i.e., $\mathcal{Q}_n$ embeds into $\mathcal{Q}_2$. □

6 Conjugacy

The conjugacy relation in groups has two natural generalizations to monoids, which, when considered in $\mathcal{Q}$, we determine in this section.

Definition 6.1. *Let M be a monoid and $p, q \in M$. Then p and q are* conjugate*, which we denote by $p \approx q$, if there exists $x \in M$ such that $px = xq$. Furthermore, p and q are* transposed*, which we denote by $p \sim q$, if there are $x, y \in M$ with $p = xy$ and $q = yx$. Moreover, $\stackrel{*}{\sim}$ is the transitive and reflexive closure of $\sim$.*

Observe that $\approx$ is reflexive and transitive whereas $\sim$ is and symmetric, and $\sim \subseteq \approx$. If M is actually a group, then both relations coincide and are equivalence relations, called conjugacy. The same is true for free monoids [Lot83, Prop. 1.3.4] and even for special monoids [Zha91], but there are monoids where none of this holds. In this section, we prove for the monoid $\mathcal{Q}$ that $\approx$ is the transitive and reflexive closure $\stackrel{*}{\sim}$ of $\sim$. Moreover, we give a simple (polynomial-time) characterization of when $p \approx q$ holds.

Notice that the relation $\sim$ on $\mathcal{Q}$ is self-dual in the following sense: Let $p, q \in \mathcal{Q}$ with $p \sim q$. Then there are $x, y \in \mathcal{Q}$ such that $p = xy$ and $q = yx$. This implies

$\delta(p) = \delta(y)\delta(x)$ and $\delta(q) = \delta(x)\delta(y)$, i.e., $\delta(p) \sim \delta(q)$. Conversely, $\delta(p) \sim \delta(q)$ also implies $p \sim q$ because δ is an involution. Consequently, $\overset{*}{\sim}$ is self-dual in the same sense as well.

Example 6.2. Let $u, v, w \in A^*$. Then $\overline{u}\,\langle v, \overline{v}\rangle\, w \equiv \overline{u}v\overline{v}w \equiv v\overline{u}\overline{v}w$. Consequently, $\mathcal{Q} = \eta(A^*\overline{A}^*A^*)$ and dually $\mathcal{Q} = \eta(\overline{A}^*A^*\overline{A}^*)$. Furthermore, $v\overline{u}\overline{v}w \overset{*}{\sim} \overline{u}\overline{v}wv$. Hence, for every $q \in \mathcal{Q}$, there exists $q' \in \eta(\overline{A}^*A^*)$ with $q \overset{*}{\sim} q'$, i.e., $\mathcal{Q}$ is the closure of $\eta(\overline{A}^*A^*)$ under transposition.

Lemma 6.3. *Let $x, y \in A^*$ and $a \in A$. If $x \neq ya$, then $[\overline{x}ya] \overset{*}{\sim} [\overline{x}ay]$.*

Proof. If $x = \varepsilon$, we have $[\overline{x}ya] = [y][a] \sim [a][y] = [\overline{x}ay]$. Hence, let $x = ub$ with $u \in A^*$ and $b \in A$. If $b \neq a$, then

$$[\overline{u}\overline{b}\,ya] \sim [a\overline{u}\overline{b}y] = [\overline{u}a\overline{b}y] = [\overline{u}\overline{b}ay]\,.$$

Henceforth, assume $b = a$. Thus, $x = ua$ and consequently $x \neq ya$ implies $u \neq y$. With $w = \mathsf{nf}(y\overline{u})$, we have

$$[\overline{x}ya] = [\overline{u}\overline{a}ya] \sim [ya\overline{u}\overline{a}] = [y\overline{u}a\overline{a}] = [wa\overline{a}]\,.$$

Notice that w cannot start with a write symbol and end with a read symbol at the same time, because this would imply $w \in \{\, a\overline{a} \mid a \in A \,\}^*$ and hence $u = y$. In case that w starts with a read symbol, we have

$$[wa\overline{a}] \sim [a\overline{a}w] = [\overline{a}aw] = [\overline{a}ay\overline{u}] \sim [\overline{u}\overline{a}ay] = [\overline{x}ay]\,.$$

In the other case, i.e., if w ends with a write symbol, we obtain

$$[wa\overline{a}] = [w\overline{a}a] = [y\overline{u}\overline{a}a] \sim [\overline{u}\overline{a}ay] = [\overline{x}ay]\,. \qquad \square$$

Lemma 6.4. *For $x, y \in A^*$ and $a \in A$, we have (1) $[\overline{x}ya] \overset{*}{\sim} [\overline{x}ay]$ and (2) $[\overline{x}\overline{a}y] \overset{*}{\sim} [\overline{a}\overline{x}y]$.*

Proof. We show claim (1) first. The case $x \neq ya$ was treated in Lemma 6.3 and we may therefore assume $x = ya$. Let $u = a_1a_2\ldots a_k$ with $a_1, \ldots, a_k \in A$ be the shortest nonempty prefix of x such that $x = vu = uv$ for the complementary suffix $v \in A^*$.

Then $x \neq a_{\ell+1}a_{\ell+2}\ldots a_k\, v\, a_1a_2\ldots a_\ell$ for all $1 \leq \ell < k$ and hence, applying Lemma 6.3 $k-1$ times, we get

$$\begin{aligned}
[\overline{x}ay] &= [\overline{x}a_kva_1a_2\ldots a_{k-1}] \\
&\overset{*}{\sim} [\overline{x}a_{k-1}a_kva_1a_2\ldots a_{k-2}] \\
&\overset{*}{\sim} [\overline{x}a_{k-2}a_{k-1}a_kva_1a_2\ldots a_{k-3}] \\
&\;\;\vdots \\
&\overset{*}{\sim} [\overline{x}a_1\ldots a_kv] = [\overline{x}ya]\,.
\end{aligned}$$

Concerning the claim (2), we first observe that

$$\delta([\overline{x}a\overline{y}]) = \left[\overline{y^R}ax^R\right] \overset{*}{\sim} \left[\overline{y^R}x^Ra\right] = \delta([\overline{a}\overline{x}y])\,.$$

Since $\sim$ is self-dual, we may conclude $[\overline{x}a\overline{y}] \overset{*}{\sim} [\overline{a}\overline{x}y]$. $\square$

The announced description of $\approx$ is a characterization in terms of the projections of the elements.

Theorem 6.5. *For any $p, q \in \mathcal{Q}$, the following are equivalent:*

(1) $p \overset{*}{\sim} q$.
(2) $p \approx q$.
(3) $q \approx p$.
(4) $\pi(p) \sim \pi(q)$ and $\overline{\pi}(p) \sim \overline{\pi}(q)$.

Proof. If $p \sim q$ with $p = rs$ and $q = sr$, then $pr = rsr = rq$ and hence $p \approx q$. Since $\approx$ is transitive, this ensures "(1)⇒(2)".

In order to show "(2)⇒(4)", suppose $px = xq$. Then we have $\pi(p)\pi(x) = \pi(x)\pi(q)$ and $\overline{\pi}(p)\overline{\pi}(x) = \overline{\pi}(x)\overline{\pi}(q)$. Since $\sim$ and $\approx$ coincide on the free monoid, this implies $\pi(p) \sim \pi(q)$ and $\overline{\pi}(p) \sim \overline{\pi}(q)$ and therefore (4).

Next, we prove "(4)⇒(1)". So assume $\pi(p) \sim \pi(q)$ and $\overline{\pi}(p) \sim \overline{\pi}(q)$. There are unique words $r, s, t, u, v, w \in A^*$ with $p = [\overline{r}\langle s, \overline{s}\rangle t]$ and $q = [\overline{u}\langle v, \overline{v}\rangle w]$. Note that $ts \sim st = \pi(p) \sim \pi(q) = vw \sim wv$ and $rs = \overline{\pi}(p) \sim \overline{\pi}(q) = uv$. Then we get

$$\begin{aligned}
p &= [\overline{r}\langle s, \overline{s}\rangle t] \\
&= [s\overline{r}\overline{s}t] && ([s]\cdot[\overline{r}\overline{s}] = [\overline{r}\langle s, \overline{s}\rangle] \text{ by Theorem 5.4}) \\
&\sim [\overline{r}\overline{s}ts] \\
&\overset{*}{\sim} [\overline{r}\overline{s}wv] && (ts \sim wv \text{ and repeated application of Lemma 6.4(1)}) \\
&\overset{*}{\sim} [\overline{u}\overline{v}wv] && (rs \sim uv \text{ and repeated application of Lemma 6.4(2)}) \\
&\sim [v\overline{u}\overline{v}w] \\
&= [\overline{u}\langle v, \overline{v}\rangle w] && ([v]\cdot[\overline{u}\overline{v}] = [\overline{u}\langle v, \overline{v}\rangle] \text{ by Theorem 5.4}) \\
&= q\,.
\end{aligned}$$

Thus, we proved the equivalence of (1), (2), and (4). It follows in particular that $\approx$ is symmetric. Hence, (2) and (3) are equivalent as well. $\square$

Given two words $u, v \in \Sigma^*$, one can decide in quadratic time whether $\pi(u) \sim \pi(v)$ and $\overline{\pi}(u) \sim \overline{\pi}(v)$. Consequently, it is decidable in polynomial time whether $[u] \approx [v]$ holds.

Open question. Is there some number $k \in \mathbb{N}$ such that $p \overset{*}{\sim} q$ if and only if $p \overset{k}{\sim} q$?

7 Conjugators

Definition 7.1. *Let M be a monoid and $x, y \in M$. An element $z \in M$ is a* conjugator *of x and y if $xz = zy$. The set of all conjugators of x and y is denoted*

$$C_M(x, y) = \{ z \in M \mid xz = zy \} .$$

Suppose that M is a free monoid A^* and consider $x, y \in A^*$. It is well-known that $z \in A^*$ is a conjugator of x and y precisely if there are $u, v \in A^*$ such that $x = uv$, $y = vu$, and $z \in u(vu)^*$. Consequently, $C_{A^*}(x, y)$ is a finite union of sets of the form $u(vu)^*$ and hence regular. In contrast, Observation 7.2 and Theorem 7.3 demonstrate that in the monoid $\mathcal{Q}$, sets of conjugators are always rational, but in general not recognizable.

Observation 7.2 *Let $a \in A$. The set $C_{\mathcal{Q}}([\overline{a}], [\overline{a}])$ is not recognizable.*

Proof. We show the claim by establishing the equation

$$\eta^{-1}\left(C_{\mathcal{Q}}([\overline{a}], [\overline{a}])\right) \cap a^*\overline{a}^* = \left\{ a^k\overline{a}^\ell \;\middle|\; k, \ell \in \mathbb{N}, k \leq \ell \right\} .$$

To this end, consider $k, \ell \in \mathbb{N}$ and let $z = \left[a^k\overline{a}^\ell\right]$. On the one hand, if $k \leq \ell$, then

$$\mathsf{nf}([\overline{a}]\, z) = \overline{a}^{\ell+1-k}\ (a\overline{a})^k = \mathsf{nf}(z\, [\overline{a}]) ,$$

i.e., $z \in C_{\mathcal{Q}}([\overline{a}], [\overline{a}])$. On the other hand, if $k > \ell$, then

$$\mathsf{nf}([\overline{a}]\, z) = \overline{a}(a\overline{a})^\ell a^{k-\ell} \neq (a\overline{a})^{\ell+1} a^{k-\ell-1} = \mathsf{nf}(z\, [\overline{a}]) ,$$

i.e., $z \notin C_{\mathcal{Q}}([\overline{a}], [\overline{a}])$. ◻

Theorem 7.3. *Let $x, y \in \mathcal{Q}$. Then the set $C_{\mathcal{Q}}(x, y)$ is rational.*

The proof needs some preparatory lemmas and follows at the end of this section. Throughout, we fix two elements $x, y \in \mathcal{Q}$ as well as their normal forms $\mathsf{nf}(x) = \overline{x_1}\, \langle x_2, \overline{x_2} \rangle\, x_3$ and $\mathsf{nf}(y) = \overline{y_1}\, \langle y_2, \overline{y_2} \rangle\, y_3$. Applying the projections π and $\overline{\pi}$ to the equation $xz = zy$ for any $z \in C_{\mathcal{Q}}(x, y)$ yields that $\pi(z)$ is a conjugator of $\pi(x)$ and $\pi(y)$ as well as that $\overline{\pi}(z)$ is a conjugator of $\overline{\pi}(x)$ and $\overline{\pi}(y)$ in the free monoid A^*. Thus, the set

$$D(x, y) = \{ z \in \mathcal{Q} \mid \pi(xz) = \pi(zy)\ \&\ \overline{\pi}(xz) = \overline{\pi}(zy) \} \supseteq C_{\mathcal{Q}}(x, y)$$

can be regarded as an overestimation of $C_{\mathcal{Q}}(x, y)$. Recall that any $q \in \mathcal{Q}$ is completely determined by $\pi(q)$, $\overline{\pi}(q)$, and $\mathsf{ow}(q)$. Thus, $z \in D(x, y)$ satisfies $z \in C_{\mathcal{Q}}(x, y)$ if and only if $\mathsf{ow}(xz) = \mathsf{ow}(zy)$. The proof of Theorem 7.3 basically exploits this observation in combination with the fact that the set $D(x, y)$ can be rephrased as

$$D(x, y) = \pi^{-1}\big(C_{A^*}(\pi(x), \pi(y))\big) \cap \overline{\pi}^{-1}\big(C_{A^*}(\overline{\pi}(x), \overline{\pi}(y))\big)$$

and is hence recognizable.

Lemma 7.4. *Every $z \in D(x,y)$ satisfies $0 \le \mathsf{ow}(xz) - \mathsf{ow}(z) \le |\pi(x)|$.*

Proof. Let $\mathsf{nf}(z) = \overline{z_1}\,\langle z_2, \overline{z_2}\rangle\, z_3$. By Theorem 5.4, we have

$$\mathsf{ow}(xz) = |\mathsf{ol}(x_2 z_1 z_2, x_2 x_3 z_2)| \le |x_2 x_3 z_2| = |\pi(x)| + \mathsf{ow}(z)\,.$$

This proves the second inequation.

Since $\pi(z) \in C_{A^*}(\pi(x), \pi(y))$, we can apply the characterization of conjugators in free monoids and write $\pi(x) = uv$ and $z_2 z_3 = \pi(z) = (uv)^k u$ for some $u, v \in A^*$ and $k \in \mathbb{N}$. Hence, $z_2 = (uv)^\ell w$ for some prefix w of uv and $\ell \in \mathbb{N}$. Thus, z_2 is a prefix of $x_2 x_3 z_2 = (uv)^{\ell+1} w$ as well as a suffix of $x_2 z_1 z_2$. Again by Theorem 5.4, this implies $\mathsf{ow}(xz) \ge |z_2| = \mathsf{ow}(z)$, i.e., the first inequation. $\square$

Lemma 7.5. *For every $k \in \mathbb{N}$, the following set is regular:*

$$G_k = \{\, \mathsf{nf}(z) \mid z \in D(x,y), \mathsf{ow}(xz) - \mathsf{ow}(z) \ge k \,\}\ .$$

Proof. Consider some $z \in D(x,y)$ and let $\mathsf{nf}(z) = \overline{z_1}\,\langle z_2, \overline{z_2}\rangle\, z_3$ be its normal form. Due to Theorem 5.4, we have $\mathsf{ow}(xz) \ge \mathsf{ow}(z) + k$ if and only if there is some $w \in A^*$ with $|w| \ge \mathsf{ow}(z) + k$ that is a suffix of $x_2 z_1 z_2$ as well as a prefix of $x_2 x_3 z_2$. Since $\mathsf{ow}(z) = |z_2|$, this is true precisely if there is some suffix $u \in A^{\ge k}$ of $x_2 z_1$ such that $u z_2$ is a prefix of $x_2 x_3 z_2$. According to Lemma 7.4, any such u also satisfies $|u| \le |\pi(x)|$. Altogether, this amounts to

$$G_k = \eta^{-1}\left(D(x,y)\right) \cap \bigcup_{\substack{u \in A^* \\ k \le |u| \le |\pi(x)|}} \overline{X_u}\,\phi(Y_u)\,A^*\,,$$

where $\phi\colon A^* \to \Sigma^*$ is the morphism defined by $\phi(v) = \langle v, \overline{v}\rangle$,

$$X_u = \{\ z_1 \in A^* \mid u \text{ is a suffix of } x_2 z_1 \,\}\ ,$$

and

$$Y_u = \{\, z_2 \in A^* \mid u z_2 \text{ is a prefix of } x_2 x_3 z_2 \,\}\ .$$

Since $D(x,y)$ is recognizable, it suffices to show that X_u and Y_u are regular for each $u \in A^*$ in order to prove the claim of the lemma.

Concerning X_u, observe that u is a suffix of $x_2 z_1$ if u is a suffix of z_1 or there is a factorization $u = v z_1$ of u such that v is a suffix of x_2. Thus,

$$X_u = A^* u \cup \{\ z_1 \mid v, z_1 \in A^*, u = v z_1, v \text{ is a suffix of } x_2 \,\}$$

and this set is clearly regular. Concerning Y_u, we first observe that $Y_u = \emptyset$ if u is not a prefix of $x_2 x_3$ and $Y_u = A^*$ if $u = x_2 x_3$. If u is a proper prefix of $x_2 x_3$, say $x_2 x_3 = uv$, then Y_u is the set of all $z_2 \in A^*$ such that z_2 is a prefix of $v z_2$. It is well-known that this is precisely the prefix closure of v^*. In each of these three cases, Y_u is regular. $\square$

Proof (of Theorem 7.3). Consider some $k \in \mathbb{N}$. By Lemma 7.5, the set

$$E_k = \{\ \mathsf{nf}(z)\ \mid z \in D(x,y), \mathsf{ow}(xz)\ - \mathsf{ow}(z) = k\,\} = G_k \setminus G_{k+1}$$

is regular. Our first goal is to show that the set

$$F_k = \{\ \mathsf{nf}(z)\ \mid z \in D(x,y), \mathsf{ow}(zy)\ - \mathsf{ow}(z) = k\,\}$$

is regular as well. To that end, it suffices to show that $\delta(F_k)$ is regular because δ is an involution that preserves regularity of subsets of Σ^*.

It is a matter of routine to check that $z \in D(x,y)$ holds true precisely if $\delta(z) \in D(\delta(y), \delta(x))$. Since δ preserves the overlap width and $\delta(\mathsf{nf}(z)) = \mathsf{nf}(\delta(z))$, we thus obtain

$$\delta(F_k) = \{\ \mathsf{nf}(\delta(z)) \mid \delta(z) \in D(\delta(y), \delta(x)), \mathsf{ow}(\delta(y)\delta(z)) - \mathsf{ow}(\delta(z))\ = k\,\}\ .$$

Using once more that δ is an involution, and hence surjective, yields

$$\delta(F_k) = \{\ \mathsf{nf}(z) \mid z \in D(\delta(y), \delta(x)), \mathsf{ow}(\delta(y)z) - \mathsf{ow}(z)\ = k\,\}\ .$$

Since the regularity of E_k does not depend on the specific choice of x and y, this set and hence also F_k are regular.

Recall that $z \in D(x,y)$ satisfies $z \in C_{\mathcal{Q}}(x,y)$ precisely if $\mathsf{ow}(xz) = \mathsf{ow}(zy)$. Using Lemma 7.4, we thus obtain

$$\{\,\mathsf{nf}(z) \mid z \in C_{\mathcal{Q}}(x,y)\,\} = \bigcup_{0 \le k \le |\pi(x)|} E_k \cap F_k\ .$$

Since this set is regular, $C_{\mathcal{Q}}(x,y)$ is rational. ☐

8 Rational subsets

This section studies decision problems concerning rational subsets of $\mathcal{Q}$. While most of these problems are undecidable, the uniform membership in rational subsets is NL-complete. We will also see that for rational subsets R of $\mathcal{Q}$, the set of normal forms $\mathsf{nf}(q)$ of elements q of R is not necessarily regular.

Let $w \in \Sigma^*$. Then one can show that the number of left-divisors of $[w]$ in $\mathcal{Q}$ is at most $|w|^3$. This allows to define a DFA with $|w|^3$ many states that accepts $[w] = \{u \in \Sigma^* \mid u \equiv w\}$. The following lemma strengthens this observation by showing that such a DFA can be constructed in logarithmic space.

Lemma 8.1. *From $w \in \Sigma^*$, one can construct in logarithmic space a DFA accepting $[w]$.*

Proof. Let $w = a_1 a_2 \dots a_n$. For $1 \le i \le n$ and $0 \le j \le n$, we define $w[i,j] = a_i a_{i+1} \dots a_j$, in particular $w[i,j] = \varepsilon$ if $i > j$.

Let $i, j, k, \ell \in \{0, 1, \dots, n\}$ be natural numbers. For the quadrupel $p = (i,j,k,\ell)$, we define four words $p_1, p_2, p_2', p_3 \in A^*$ setting

- $p_1 = \overline{\pi}(w[1,i])$ and $p_2 = \overline{\pi}(w[i+1,j])$ as well as
- $p'_2 = \pi(w[1,k])$ and $p_3 = \pi(w[k+1,\ell])$.

Then p is a state of the DFA if and only if

- $p_2 = p'_2$,
- $i = 0$ or $a_i \in \overline{A}$ and similarly $j = 0$ or $a_j \in \overline{A}$, and
- $k = 0$ or $a_k \in A$ and similarly $\ell = 0$ or $a_\ell \in A$.

Hence every state p of the DFA stands for a word $u_p = p_1 \langle p'_2, \overline{p_2} \rangle p_3$ in normal form.

The initial state of the DFA is $\iota = (0,0,0,0)$ such that $u_\iota = \varepsilon$. The state $p = (i,j,k,\ell)$ is accepting if $u_p \equiv w$.

Our aim is to define the transitions of the automaton in such a way that, after reading $v \in \Sigma^*$, the automaton reaches a state p with $u_p = \mathsf{nf}(v)$, provided that such a state exists. Furthermore, we want to make sure that such a state exists whenever $[v]$ is a left-divisor of $[w]$.

So let $p = (i,j,k,\ell)$ be a state and $a \in A$. To define the state reached from p after reading a, let $\ell' > \ell$ be the minimal write-position in w after ℓ. In other words, $\ell < \ell'$, $a_{\ell'} \in A$ and $w[\ell+1, \ell'-1] \in \overline{A}^*$. If there is no such ℓ' or if $a_{\ell'} \neq a$, then the DFA cannot make any a-move from state p. Otherwise, it moves to $q = (i,j,k,\ell')$. It is easily verified that this tuple is a state again since p is a state and since $a_{\ell'} = a \in A$. We have

$$\begin{aligned}
u_p a &= \overline{p_1} \langle p'_2, \overline{p_2} \rangle p_3\, a \\
&= \overline{\overline{\pi}(w[1,i])} \left\langle \pi(w[1,k]), \overline{\overline{\pi}(w[i+1,j])} \right\rangle \pi(w[k+1,\ell])\, a \\
&= \overline{\overline{\pi}(w[1,i])} \left\langle \pi(w[1,k]), \overline{\overline{\pi}(w[i+1,j])} \right\rangle \pi(w[k+1,\ell']) \\
&= u_q\,.
\end{aligned}$$

We next define which state is reached from p after reading $\overline{a}$. Let j' be the minimal read-position in w after j. In other words, $j < j'$, $a_{j'} \in \overline{A}$, and $w[j+1, j'-1] \in A^*$. If no such j' exists or if $a_{j'} \neq \overline{a}$, then the DFA cannot make any $\overline{a}$-move from state p. So assume j' exists with $a_{j'} = \overline{a}$. Then consider the word

$$s = \mathsf{ol}(\overline{\pi}(w[i+1,j']), \pi(w[1,\ell]))$$

which equals $\mathsf{ol}(p_2 a, p_2 p_3)$ since $\overline{\pi}(w[i+1,j']) = \overline{\pi}(w[i+1,j])a = p_2 a$. Since s is a suffix of $\overline{\pi}(w[i+1,j'])$, there exists $i \leq i' \leq j'$ with $s = \overline{\pi}(w[i'+1,j'])$. In addition, we can assume $i' = 0$ or $a_{i'} \in \overline{A}$. Similarly, since s is a prefix of $\pi(w[1,\ell])$, there exists $1 \leq k' \leq k$ with $s = \pi(w[1,k'])$ and $k' = 0$ or $a_{k'} \in A$. Now the tuple $q = (i', j', k', \ell)$ is a state of the DFA and the DFA moves from p to q when reading $\overline{a}$.

Set

$$\begin{aligned}
r &= p_2 a s^{-1} = \overline{\pi}(w[i+1,j'])\overline{\pi}(w[i'+1,j'])^{-1} = \overline{\pi}(w[i+1,i']) \quad \text{and} \\
t &= s^{-1} p_2 p_3 = \pi(w[1,k'))^{-1} \pi(w[1,\ell]) = \pi(w[k'+1,\ell))\,.
\end{aligned}$$

Then we get

$$\begin{aligned}
u_p\overline{a} &= \overline{p_1}\,\langle p_2, \overline{p_2}\rangle\, p_3 \cdot \overline{a} \\
&\equiv \overline{p_1 r}\,\langle s, \overline{s}\rangle\, t && \text{(by Theorem 5.4)} \\
&= \overline{\overline{\pi}(w[1,i'])}\,\Big\langle \pi(w[1,k']), \overline{\overline{\pi}(w[i'+1,j'])}\Big\rangle\, \pi(w[k'+1,\ell]) \\
&= u_q\,.
\end{aligned}$$

This finishes the construction of the DFA.

Now let $v \in \Sigma^*$. If there is a v-labeled path from the initial state $(0,0,0,0)$ to some state q, then by induction on $|v|$, we obtain $v \equiv u_q$ from the above calculations. In particular, any word v accepted by the DFA satisfies $v \equiv w$, i.e., $v \in [w]$.

Before proving the converse implication, let $v \in \Sigma^*$ such that $[v]$ is a left-divisor of $[w]$. Let $\mathsf{nf}(v) = \overline{v_1}\,\langle v_2, \overline{v_2}\rangle\, v_3$. Since $\overline{\pi}, \pi \colon \mathcal{Q} \to A^*$ are morphisms, v_1v_2 is a prefix of $\overline{\pi}(w)$ and v_2v_3 is a prefix of $\pi(w)$. By induction on $|v|$, one obtains that there is a v-labeled path from $(0,0,0,0)$ to some state $p = (i,j,k,\ell)$. Our observation above tells us that then $v \equiv u_p$. In particular, if $v \in [w]$, then $u_p \equiv w$, i.e., p is accepting. Thus, the DFA accepts $[w]$.

By the construction of the DFA, it is clear that a Turing machine with w on its input tape can, using logarithmic space on its work tape, write the list of all transitions on its one-way output tape. □

Theorem 8.2. *The following rational subset membership problem for $\mathcal{Q}$ is* NL-*complete:*

Input: a word $w \in \Sigma^*$ and an NFA $\mathcal{A}$ over Σ.
Question: Is there a word $v \in L(\mathcal{A})$ with $w \equiv v$?

Proof. Let $w \in \Sigma^*$ and let $\mathcal{A}$ be an NFA over Σ. Let $\mathcal{B}$ be the DFA from Lemma 8.1 that can be construced in logarithmic space.

Then there exists $v \in L(\mathcal{A})$ with $w \equiv v$ if and only if $L(\mathcal{A}) \cap [w] \neq \emptyset$ if and only if $L(\mathcal{A}) \cap L(\mathcal{B}) \neq \emptyset$. Using an on-the-fly construction of $\mathcal{B}$, this can be decided nondeterministically in logarithmic space. Hence, the problem is in NL.

Since the free monoid A^* embeds into $\mathcal{Q}$ and since the rational subset membership problem for A^* is NL-hard, we also get NL-hardness for $\mathcal{Q}$. □

In the rest of this section, we will prove some negative results on rational subsets of $\mathcal{Q}$. All these results rest on a particular embedding of the monoid $\{a,b\}^* \times \{c,d\}^*$ into $\mathcal{Q}$. This embedding is discussed in the following proposition.

Proposition 8.3. *Let $\mathcal{R} = \{[a], [ab], [\overline{b}], [\overline{abb}]\}^* \subseteq \mathcal{Q}$ denote the submonoid generated by $\{[a], [ab], [\overline{b}], [\overline{abb}]\}$.*

(1) There exists an isomorphism α from $\{a,b\}^* \times \{c,d\}^*$ onto $\mathcal{R}$ with $\alpha((a,\varepsilon)) = [a]$, $\alpha((b,\varepsilon)) = [ab]$, $\alpha((\varepsilon,c)) = [\overline{b}]$, and $\alpha((\varepsilon,d)) = [\overline{abb}]$.
(2) If $S \subseteq \mathcal{R}$ is recognizable in $\mathcal{R}$, then it is recognizable in $\mathcal{Q}$.

Proof. Let $\beta\colon \{a,b,c,d\}^* \to \mathcal{R}$ be the morphism defined by $\beta(a) = [a]$, $\beta(b) = [ab]$, $\beta(c) = [\overline{b}]$, and $\beta(d) = [\overline{abb}]$. Note that β is surjective.

Furthermore, note that

$$\{a,b\}^* \times \{c,d\}^* \cong \{a,b,c,d\}^*/\{ac = ca, bc = cb, ad = da, bd = db\}\,.$$

Theorem 5.4 implies in particular

$$\begin{aligned}
\beta(ac) &= [a\overline{b}] = [\overline{b}a] = \beta(ca)\,,\\
\beta(bc) &= [ab\overline{b}] = [\overline{b}ab] = \beta(cb)\,,\\
\beta(ad) &= [a\overline{abb}] = [\overline{abb}a] = \beta(da)\,,\ \text{and}\\
\beta(bd) &= [ab\overline{abb}] = [\overline{abb}ab] = \beta(db)
\end{aligned}$$

since $\mathsf{ol}(\beta(x),\beta(y)) = \varepsilon$ for all $(x,y) \in \{a,b\} \times \{c,d\}$.

Hence we can lift β to a morphism $\alpha\colon \{a,b\}^* \times \{c,d\}^* \to \mathcal{R}$. The surjectivity of α follows from that of β.

Note that $\pi(\alpha(u,v)) = \pi(\alpha(u,\varepsilon))$ for $(u,v) \in \{a,b\}^* \times \{c,d\}^*$ and observe that the map $\{a,b\}^* \to A^*$, $u \mapsto \pi(\alpha(u,\varepsilon))$, is injective. Suppose $\alpha(u,v) = \alpha(u',v')$. Then

$$\pi(\alpha(u,\varepsilon)) = \pi(\alpha(u,v)) = \pi(\alpha(u',v')) = \pi(\alpha(u',\varepsilon))$$

and thus $u = u'$. Analogously, we deduce $v = v'$. Hence, α is injective and therefore an isomorphism as required.

Finally let $S \subseteq \mathcal{R}$ be a recognizable subset of $\mathcal{R}$. Then the subset $\alpha^{-1}(S) \subseteq \{a,b\}^* \times \{c,d\}^*$ is recognizable. By Mezei's theorem, there exist regular languages $U_i \subseteq \{a,b\}^*$ and $V_i \subseteq \{c,d\}^*$ with $\alpha^{-1}(S) = \bigcup_{1\le i\le n} U_i \times V_i$. Define the morphism $g\colon \{a,b\}^* \to A^*$ with $g(a) = a$ and $g(b) = ab$ as well as the morphism $h\colon \{c,d\}^* \to A^*$ with $h(c) = b$ and $h(d) = abb$. Since morphisms between free monoids preserve regularity, the languages $g(U_i), h(V_i) \subseteq A^*$ are regular. Therefore, $\pi^{-1}(g(U_i))$ and $\overline{\pi}^{-1}(h(V_i))$ are recognizable in $\mathcal{Q}$. Hence also

$$\bigcup_{1\le i\le n} \pi^{-1}(g(U_i)) \cap \overline{\pi}^{-1}(h(U_i))$$

is recognizable in $\mathcal{Q}$. This set equals S. $\square$

Open question. Since the monoid $\{a,b\}^* \times \{c,d\}^*$ embeds into $\mathcal{Q}$, so does the monoid $(\mathbb{N}^2,+)$. It is not known whether $\mathbb{N}^3$ can be embedded into $\mathcal{Q}$.

Theorem 8.4. *(1) The set of rational subsets of $\mathcal{Q}$ is not closed under intersection.*

(2) The emptiness of the intersection of two rational subsets of $\mathcal{Q}$ is undecidable.

(3) The universality of a rational subset of $\mathcal{Q}$ is undecidable.
Consequently, inclusion and equality of rational subsets are undecidable.

(4) The recognizability of a rational subset of $\mathcal{Q}$ is undecidable.

Proof. Throughout this proof, let α be the isomorphism from Prop. 8.3.

(1) Consider the rational relations

$$R_1 = \{(a^m, c^m d^n) \mid m, n \geq 1\} \text{ and } R_2 = \{(a^m, c^n d^m) \mid m, n \geq 1\}\,.$$

Then the sets

$$\alpha(R_1) = \{x \in \mathcal{Q} \mid \exists m, n \geq 1\colon \pi(w) = a^m, \overline{\pi}(w) = b^m (abb)^n\} \quad \text{and}$$
$$\alpha(R_2) = \{x \in \mathcal{Q} \mid \exists m, n \geq 1\colon \pi(w) = a^m, \overline{\pi}(w) = b^n (abb)^m\}$$

are rational in $\mathcal{Q}$. Suppose their intersection $\alpha(R_1) \cap \alpha(R_2)$ is rational. Then there exists a regular language $S \subseteq \Sigma^*$ with

$$\alpha(R_1) \cap \alpha(R_2) = \eta(S)\,.$$

It follows that the language $\overline{\pi}(S) \subseteq A^*$ is regular. But this set equals the language $\{b^m (abb)^m \mid m \geq 1\} \subseteq \Sigma^*$ which is not regular.

(2) Let $R_1, R_2 \subseteq \{a,b\}^* \times \{c,d\}^*$ be rational. Then $\alpha(R_1)$ and $\alpha(R_2)$ are rational and, since α is an isomorphism, $\alpha(R_1) \cap \alpha(R_2) = \alpha(R_1 \cap R_2)$. Consequently, $\alpha(R_1) \cap \alpha(R_2) = \emptyset$ if and only if $R_1 \cap R_2 = \emptyset$. But this latter question is undecidable [Ber79, Theorem 8.4(i)].

(3) Let $S \subseteq \{a,b\}^* \times \{c,d\}^*$ be rational. Then $\alpha(S)$ is rational. Due to Prop. 8.3(2), the set $\mathcal{R}$ is recognizable in $\mathcal{Q}$. Therefore, $\mathcal{Q} \setminus \mathcal{R}$ is recognizable and hence rational because $\mathcal{Q}$ is finitely generated. Consequently, $\alpha(S) \cup (\mathcal{Q} \setminus \mathcal{R})$ is rational as well. This rational set equals $\mathcal{Q}$ if and only if $\alpha(S) = \mathcal{R}$, i.e., $S = \{a,b\}^* \times \{c,d\}^*$. But this latter question is undecidable by [Ber79, Theorem 8.4(iv)].

(4) Let $S \subseteq \{a,b\}^* \times \{c,d\}^*$ be rational. Then $\alpha(S)$ is rational. By Prop. 8.3(2), $\alpha(S)$ is recognizable in $\mathcal{Q}$ if and only if it is recognizably in $\mathcal{R}$. But this is the case if and only if S is recognizable in $\{a,b\}^* \times \{c,d\}^*$. This latter question is undecidable by [Ber79, Theorem 8.4(vi)]. ☐

Suppose a monoid M is presented by a convergent semi-Thue system that is *monadic*, meaning that every right-hand side is either a single letter or the empty word. Then it is well-known that a subset $R \subseteq M$ is rational if and only if the set of normal forms of members of R is regular [BO93]. This is not the case in our setting. For instance, the set $\{[a\overline{b}]\}^*$ is rational, but its set of normal forms is the non-regular language $\{\overline{b}^n a^n \mid n \geq 0\}$. Using Theorem 8.4, we can generalize this example.

A *cross-section* is a subset $S \subseteq \Sigma^*$ such that $\eta|_S\colon S \to \mathcal{Q}$ is a bijection. An example of a cross-section is the set of normal forms.

Corollary 8.5. *There is no cross-section $S \subseteq \Sigma^*$ such that we have: $R \subseteq \mathcal{Q}$ is rational if and only if $\eta^{-1}(R) \cap S$ is regular.*

Proof. Suppose there were such a cross-section $S \subseteq \Sigma^*$. Then, for rational subsets $R_1, R_2 \subseteq \mathcal{Q}$, we have

$$\begin{aligned}(\eta^{-1}(R_1) \cap S) \cap (\eta^{-1}(R_2) \cap S) &= (\eta^{-1}(R_1) \cap \eta^{-1}(R_2)) \cap S\\ &= \eta^{-1}(R_1 \cap R_2) \cap S.\end{aligned}$$

Since $\eta^{-1}(R_i) \cap S$ is regular for $i = 1, 2$, so is $\eta^{-1}(R_1 \cap R_2) \cap S$. Hence, $R_1 \cap R_2$ is rational as well. This means, the class of rational subsets of $\mathcal{Q}$ is closed under intersection, which contradicts Theorem 8.4(1). $\square$

9 Recognizable subsets

In this section, we aim to describe the recognizable subsets of $\mathcal{Q}$. Clearly, sets of the form $\pi^{-1}(L)$ or $\overline{\pi}^{-1}(L)$ for some regular $L \subseteq A^*$ as well as Boolean combinations thereof are recognizable. This does not suffice to produce all recognizable subsets: for instance, the singleton set $\{[\overline{a}a]\}$ is recognizable but any Boolean combination of inverse projections containing $[\overline{a}a]$ also includes $[a\overline{a}]$. However, we will see in the main result of this section, namely Theorem 9.4, that incorporating certain sets that can impose a simple restriction on relative positions of write and read symbols suffices to generate the recognizable sets as a Boolean algebra.

We also describe the recognizable subsets of $\mathcal{Q}$ from another perspective. In order to prove that a certain subset $S \subseteq \mathcal{Q}$ is recognizable, one needs to exhibit a finite automaton for the set $\eta^{-1}(S)$. However, it seems complicated to prove that an automaton accepts precisely all representatives of a given set. This raises the question of whether it is sufficient to exhibit an automaton for a small subset $K \subseteq \Sigma^*$ of all representatives. Clearly, if K is regular, then regularity of $\eta^{-1}(S) \cap K$ is necessary for S to be recognizable. The question is therefore: Is there a (simple) regular K such that regularity of $\eta^{-1}(S) \cap K$ is also sufficient?

A natural candidate for such a K would be the set of normal forms, but as we saw in Section 7, $\mathcal{Q}$ possesses non-recognizable subsets whose set of normal forms is regular (the proof of Theorem 7.3 yields regularity of the set of normal forms of $C_{\mathcal{Q}}(x, y)$, which is not recognizable in general). However, Theorem 9.4 will tell us that $K = A^*\overline{A}^*A^*$ (or, dually, $K = \overline{A}^*A^*\overline{A}^*$) has our desired property. We will also see that we cannot simplify further to $K = A^*\overline{A}^*$ or $K = \overline{A}^*A^*$ or even the union $K = A^*\overline{A}^* \cup \overline{A}^*A^*$.

Recall Observation 4.5, which states that any $q \in \mathcal{Q}$ is completely determined by $\pi(q)$, $\overline{\pi}(q)$, and $\mathsf{ow}(q)$. Consequently, it would seem natural to incorporate sets which restrict the overlap width. Unfortunately, the overlap width is not a recognizable property in the following sense:

Observation 9.1 *Let $k \in \mathbb{N}$. The set of all $q \in \mathcal{Q}$ with $\mathsf{ow}(q) = k$ is not recognizable.*

Proof. It suffices to show that the set

$$L_k = \{\, w \in \Sigma^* \mid \mathsf{ow}(w) = k \,\}$$

is not regular. For the sake of a contradiction, suppose there was an NFA $\mathcal{A}$ recognizing L_k. Let $n \geq k$ be an upper bound on the number of states of $\mathcal{A}$. Consider the word $w = a^n b a^k \overline{a}^{n-1}\overline{b}\overline{a}^k$. Since $\mathsf{nf}(w) = \overline{a}^{n-1}\overline{b}\left\langle a^k, \overline{a}^k \right\rangle a^{n-k} b a^k$, we have $\mathsf{ow}(w) = k$, i.e., $w \in L_k$. Therefore, $\mathcal{A}$ accepts w. Using a pumping argument, we obtain $\ell \le n - 1$ such that $\mathcal{A}$ also accepts $w' = a^\ell b a^k \overline{a}^{n-1} \overline{b} \overline{a}^k$. However, $\mathsf{nf}(w') = \overline{a}^{n-1-\ell} \left\langle a^\ell b a^k, \overline{a}^\ell \overline{b} \overline{a}^k \right\rangle$ implies $\mathsf{ow}(w) = \ell + 1 + k > k$ and hence $w \notin L_k$. Contradiction. $\square$

In fact, the proof above also shows that the set of all $q \in \mathcal{Q}$ with $\mathsf{ow}(q) \le k$ is not recognizable for any $k \in \mathbb{N}$. Thus, the set of all $q \in \mathcal{Q}$ with $\mathsf{ow}(q) > k$ is not recognizable either.

Nevertheless, the definition below provides a slight variation of this idea conducing to our purpose. To simplify notation, we say two elements $p, q \in \mathcal{Q}$ *have the same projections* and write $p \sim_\pi q$ if $\pi(p) = \pi(q)$ and $\overline{\pi}(p) = \overline{\pi}(q)$.

Definition 9.2. *Let $k \in \mathbb{N}$. The set $\Omega_k \subseteq \mathcal{Q}$ is given by*

$$\Omega_k = \{\, q \in \mathcal{Q} \mid \forall p \in \mathcal{Q} \colon p \sim_\pi q \;\&\; \mathsf{ow}(q) \le \mathsf{ow}(p) \le k \implies p = q \,\} \;.$$

Observe that $\mathcal{Q} = \Omega_0 \supseteq \Omega_1 \supseteq \Omega_2 \supseteq \ldots$. Intuitively, for fixed projections $\pi(q)$ and $\overline{\pi}(q)$ the set Ω_k contains all q with $\mathsf{ow}(q) \ge k$ as well as the unique q with maximal $\mathsf{ow}(q) \le k$. From this perspective, the set Ω_k is similar to the set in Observation 9.1 but uses an overestimation of the overlap width instead of the overlap width itself.

Example.

(1) The queue action $q = [\overline{a}\overline{b}a\overline{a}ba]$ satisfies $\mathsf{ow}(q) = 1$ and hence $q \in \Omega_1$. The only $p \in \mathcal{Q}$ with $p \sim_\pi q$ and $\mathsf{ow}(p) \ge \mathsf{ow}(q)$ is $p = [a\overline{a}b\overline{b}a\overline{a}]$. Since $\mathsf{ow}(p) = 3$, this implies $q \in \Omega_2$ but $q \notin \Omega_3$.
(2) For every $k \ge 1$, we have $[(\overline{a}a)^k] = \left[\overline{a} \left\langle a^{k-1}, \overline{a^{k-1}} \right\rangle a\right] \in \Omega_{k-1} \setminus \Omega_k$.
(3) All queue actions of the form $q = [u\overline{v}]$ with $u, v \in A^*$ satisfy $q \in \Omega_k$ for every $k \in \mathbb{N}$ since q realizes the maximal overlap width among all $p \in \mathcal{Q}$ with $\pi(p) = u$ and $\overline{\pi}(p) = v$.

The following observation is to the sets Ω_k as Observation 4.5 is to the overlap width and provides some more motivation for defining the sets Ω_k.

Observation 9.3 *Every $q \in \mathcal{Q}$ is completely described by $\pi(q)$, $\overline{\pi}(q)$, and the supremum over all $k \in \mathbb{N}$ with $q \in \Omega_k$.*

Proof. Fix $u, v \in A^*$ and consider some $q \in \mathcal{Q}$ with $\pi(q) = u$ and $\overline{\pi}(q) = v$. Let $m = \sup \{\, k \in \mathbb{N} \mid q \in \Omega_k \,\}$. Due to Observation 4.5, it suffices to provide $\mathsf{ow}(q)$ in terms of u, v, and m. To this end, let $w \in A^*$ be the longest suffix of v that is also a prefix of u and satisfies $|w| \le m$. In particular, we have $q \in \Omega_{|w|}$. We claim that $\mathsf{ow}(q) = |w|$.

First, we have $\mathsf{ow}(q) \le m$. This is trivial for $m = \infty$ and follows directly from $q \notin \Omega_{m+1}$ for $m < \infty$. Since there is a suffix of length $\mathsf{ow}(q)$ of $\overline{\pi}(q) = v$ that is also a prefix of $\pi(q) = u$ and due to the maximality of the length of w, we may conclude $\mathsf{ow}(q) \le |w|$. The choice of w further implies the existence of some $p \in \mathcal{Q}$ with $p \sim_\pi q$ and $\mathsf{ow}(p) = |w|$. From $q \in \Omega_{|w|}$ and $\mathsf{ow}(q) \le \mathsf{ow}(p) \le |w|$, we conclude $p = q$ and hence $\mathsf{ow}(q) = |w|$. $\square$

The aforementioned main result of this section characterizing the recognizable subsets of $\mathcal{Q}$ is Theorem 9.4 below.

Theorem 9.4. *For every subset $L \subseteq \mathcal{Q}$, the following are equivalent:*

(1) L is recognizable,
(2) $\eta^{-1}(L) \cap A^*\overline{A}^*A^*$ is regular,
(3) $\eta^{-1}(L) \cap \overline{A}^*A^*\overline{A}^*$ is regular,
(4) L is a Boolean combination of sets of the form $\pi^{-1}(R)$ or $\overline{\pi}^{-1}(R)$ for some regular $R \subseteq A^*$ and the sets Ω_k for $k \in \mathbb{N}$.

In light of this theorem, the question arises whether the regularity of $\eta^{-1}(L) \cap \overline{A}^*A^*$ or of $\eta^{-1}(L) \cap A^*\overline{A}^*$ or of both of them already suffices to conclude recognizability of L. The answer is negative, as demonstrated by the following example. The set $L = \{\, [\overline{a}^n a\overline{a}a^n] \mid n \geq 1 \,\}$ is not recognizable, since the set of its normal forms is not regular. However, both of the sets $\eta^{-1}(L) \cap \overline{A}^*A^*$ and $\eta^{-1}(L) \cap A^*\overline{A}^*$ are empty and hence regular.

The implication "(1)⇒(2)" is trivial. Throughout the rest of this section, we call subsets $L \subseteq \mathcal{Q}$ satisfying condition (2) above *wrw-recognizable*. The motivation behind wrw-recognizability is a follows: Consider a queue action $q \in \mathcal{Q}$ and let $\mathsf{nf}(q) = \overline{u}\,\langle v, \overline{v}\rangle\, w$. Lemma 5.2 and Corollary 3.8 yield $\overline{u}\,\langle v, \overline{v}\rangle\, w \equiv \overline{u}v\overline{v}w \equiv v\overline{u}\overline{v}w$, i.e., $q = [v\overline{u}\overline{v}w]$. Thus, we have $q \in L$ if and only if $\eta^{-1}(L) \cap A^*\overline{A}^*A^*$ contains at least one representative of q, although it might include even more than one representative. Finally, notice that condition (3) is dual to condition (2).

A complete proof of Theorem 9.4 follows at the end of this section. Our first step into this direction is to demonstrate the implication "(4)⇒(1)". Basically, we only have to show that Ω_k is recognizable for each $k \in \mathbb{N}$ (see Proposition 9.7). To this end, we say that a word $w \in \Sigma^*$ is *k-shuffled* if it contains at least k write and k read symbols, respectively, and for each $i = 1, \ldots, k$ the i-th write symbol of w appears before the i-th of the last k read symbols of w. We need the following relationship between the overlap width and k-shuffledness.

Lemma 9.5. *Let $k \in \mathbb{N}$, $w \in \Sigma^*$, and $u \in A^k$ a prefix of $\pi(w)$ as well as a suffix of $\overline{\pi}(w)$. Then w is k-shuffled if and only if $\mathsf{ow}(w) \geq k$.*

Proof. We show both claims by induction on $n \in \mathbb{N}$ with $w \xrightarrow{n} \mathsf{nf}(w)$. If $n = 0$, then w is in normal form and the claim is obvious.

Henceforth, we assume $n > 0$. Let $w' \in \Sigma^*$ with $w \to w' \xrightarrow{n-1} \mathsf{nf}(w)$. In particular, there are $x, y \in \Sigma^*$ and $a, b \in A$ such that $w = xa\overline{b}y$ and $w' = x\overline{b}ay$. By the induction hypothesis, the claim holds for w'. As we have $\pi(w) = \pi(w')$, $\overline{\pi}(w) = \overline{\pi}(w')$, and $\mathsf{ow}(w) = \mathsf{ow}(w')$, it suffices to show that w is k-shuffled if and only if w' is k-shuffled. The "if"-part is easy to check even without using u.

The claim of the "only if"-part is trivial unless a is among the first k write symbols of w, say the i-th of them, and $\overline{b}$ among the last k read symbols of w, say the j-th of them. If $i > j$, then the i-th of the last k read symbols of w is contained in y and the j-th write symbol of w is contained in x. Thus, w' is also

k-shuffled. We cannot have $i < j$, because then the j-th write symbol of w would have to appear after a but before $\overline{b}$.

Finally, we show that $i = j$ is also impossible. According to the exact rule used in $w \to w'$, we distinguish three cases. If $a = b$ and the rule was $ca\overline{b} \to c\overline{b}a$ for some $c \in A$, then $i > 1$ and the $(i-1)$-th of the last k read symbols of w would have to appear after x but before $\overline{b}$. Dually, if $a = b$ and the rule was $a\overline{b}\overline{c} \to \overline{b}a\overline{c}$ for some $c \in A$, then $j < k$ and the $(j+1)$-th write symbol would have to appear after a but before y. If $a \neq b$ and the rule was $a\overline{b} \to \overline{b}a$, this would contradict the fact the i-th write symbol of w as well as the i-th of the last k read symbols of w coincide with the i-th symbol of u. □

Lemma 9.6. *For each $k \in \mathbb{N}$, we have*

$$\eta^{-1}(\Omega_k) = \left\{ w \in \Sigma^* \;\middle|\; \forall u \in A^{\leq k} \colon \begin{array}{c} u \textit{ prefixes } \pi(w) \;\& \\ u \textit{ suffixes } \overline{\pi}(w) \end{array} \implies w \textit{ is } |u|\textit{-shuffled} \right\}.$$

Proof. Denote the set on the right hand side by Z_k. First, suppose $w \in \eta^{-1}(\Omega_k)$ and consider some $u \in A^{\leq k}$ that is a prefix of $\pi(w)$ as well as a suffix of $\overline{\pi}(w)$. Let $x, y \in A^*$ such that $\pi(w) = uy$ and $\overline{\pi}(w) = xu$. The queue action $p = [\overline{x}\,\langle u, \overline{u}\rangle\, y]$ satisfies $p \sim_\pi [w]$ and $\mathsf{ow}(p) = |u| \leq k$. Since $[w] \in \Omega_k$, this implies $|u| = \mathsf{ow}(p) \leq \mathsf{ow}(w)$. By Lemma 9.5, we obtain that w is $|u|$-shuffled and hence $w \in Z_k$.

Now, assume $w \in Z_k$ and consider some $p \in \mathcal{Q}$ with $p \sim_\pi [w]$ and $\mathsf{ow}(w) \leq \mathsf{ow}(p) \leq k$. Let $\mathsf{nf}(p) = \overline{x}\,\langle u, \overline{u}\rangle\, y$. Then $|u| = \mathsf{ow}(p) \leq k$ and u is a prefix of $\pi(p) = \pi(w)$ as well as a suffix of $\overline{\pi}(p) = \overline{\pi}(w)$. Since $w \in Z_k$, this implies that w is $|u|$-shuffled. From Lemma 9.5, we finally conclude $\mathsf{ow}(w) \geq |u| = \mathsf{ow}(p)$. This proves $[w] \in \Omega_k$, i.e., $w \in \eta^{-1}(\Omega_k)$. □

Proposition 9.7. *For each $k \in \mathbb{N}$, the set Ω_k is recognizable.*

Proof. It suffices to show that the set $\eta^{-1}(\Omega_k)$ is regular. For $\ell \in \mathbb{N}$, let S_ℓ denote the set of all $w \in \Sigma^*$ that are ℓ-shuffled. Lemma 9.6 translates directly into

$$\eta^{-1}(\Omega_k) = \bigcap_{u \in A^{\leq k}} \Sigma^* \setminus \big(\pi^{-1}(uA^*) \cap \overline{\pi}^{-1}(A^*u)\big) \cup S_{|u|}\,.$$

Thus, it only remains to show that all the sets S_ℓ for $\ell \leq k$ are regular. A word $w \in \Sigma^*$ is ℓ-shuffled if and only if it admits for each $i = 1, \ldots, \ell$ a factorization $w = x_i a_i y_i \overline{b_i} z_i$ with $x_i, y_i, z_i \in \Sigma^*$, $a_i, b_i \in A$, $|\pi(x_i)| = i - 1$, and $|\overline{\pi}(z_i)| = \ell - i$ (a_i is the i-th write symbol, $\overline{b_i}$ the i-th of the last ℓ read symbols). This translates directly into

$$S_\ell = \bigcap_{1 \leq i \leq \ell} \pi^{-1}(A^{i-1})\, A\, \Sigma^*\, \overline{A}\, \overline{\pi}^{-1}(A^{\ell-i})\,. \qquad \square$$

Our next step towards proving Theorem 9.4 is to establish the implication "(2)⇒(4)" (see Proposition 9.12). Again, we prepare this by a series of lemmas. Throughout, we call a subset $L \subseteq \mathcal{Q}$ *simple* if is satisfies condition (4) of Theorem 9.4. Recall that sets meeting condition (2) are called *wrw-recognizable.*

Lemma 9.8. *Let $k \in \mathbb{N}$, $q \in \Omega_k$, and $u \in A^k$ be a prefix of $\pi(q)$. Then there exists $p \in \mathcal{Q}$ such that $q = [u]\, p$.*

Proof. Let $\mathsf{nf}(q) = \overline{x}\, \langle y, \overline{y} \rangle\, z$. If u is already a prefix of y, say $y = uv$, we choose $p = [v\overline{x}\overline{y}z]$ and obtain $q = [uv\overline{x}\overline{y}z] = [u]\, p$. Now, suppose that u is not a prefix of y. Then there is a prefix v of z, say $z = vw$, such that $u = yv$. The queue action $r = [yv\overline{x}\overline{y}w]$ satisfies $r \sim_\pi q$ and $\mathsf{ow}(r) \leq |yv| = k$. Since $q \in \Omega_k$, this implies $\mathsf{ow}(r) \leq \mathsf{ow}(q) = |y|$. At the same time, $\mathsf{ow}(r) \geq |y|$ and hence $q = r$. Thus, we obtain $q = [u]\, p$ for $p = [\overline{x}\overline{y}w]$. ☐

Lemma 9.9. *Let $k \in \mathbb{N}$ and $L \subseteq \mathcal{Q}$. If L is wrw-recognizable, then the following set is simple:*

$$L \cap \pi^{-1}\left(A^{<k}\right) \cap \Omega_k\,.$$

Proof. Let $K = \eta^{-1}(L) \cap A^*\overline{A}^*A^*$ and $\phi\colon \Sigma^* \to M$ be a morphism recognizing K. We further consider the morphisms $\mu, \overline{\mu}\colon A^* \to M$ defined by $\mu(w) = \phi(w)$ and $\overline{\mu}(w) = \phi(\overline{w})$. We show the claim by establishing the equation

$$L \cap \pi^{-1}\left(A^{<k}\right) \cap \Omega_k = \bigcup_{\substack{u \in A^{<k}, m \in M \\ \mu(u)m \in \phi(K)}} \pi^{-1}(u) \cap \overline{\pi}^{-1}\left(\overline{\mu}^{-1}(m)\right) \cap \Omega_k\,.$$

Let X and Y denote the left and right hand side of this equation, respectively. Clearly, $X, Y \subseteq \pi^{-1}\left(A^{<k}\right) \cap \Omega_k$. Consider some $q \in \pi^{-1}\left(A^{<k}\right) \cap \Omega_k$. It suffices to show that $q \in X$ precisely if $q \in Y$.

To this end, let $u = \pi(q)$. Then $|u| < k$ and hence $q \in \Omega_k \subseteq \Omega_{|u|}$. Due to Lemma 9.8, there is $p \in \mathcal{Q}$ such that $q = [u]\, p$. Clearly, $\pi(p) = \varepsilon$, i.e., $p = [\overline{y}]$ for some $y \in A^*$. Notice that $q = [u\overline{y}]$. Altogether,

$$\begin{aligned} q \in X \quad &\iff \quad q = [u\overline{y}] \in L \\ &\iff \quad \phi(u\overline{y}) = \mu(u)\, \overline{\mu}\left(\overline{\pi}(q)\right) \in \phi(K) \quad \iff \quad q \in Y\,. \end{aligned}$$

☐

Lemma 9.10. *Let $k \in \mathbb{N}$ and $L \subseteq \mathcal{Q}$. If L is wrw-recognizable by a monoid with k elements, then the following set is simple:*

$$L \cap \pi^{-1}\left(A^{\geq k}\right) \cap \Omega_k\,.$$

Proof. Let K, ϕ, M, μ, and $\overline{\mu}$ be as in the proof of Lemma 9.9 and additionally assume that $|M| = k$. We show the claim by establishing the equation

$$L \cap \pi^{-1}\left(A^{\geq k}\right) \cap \Omega_k = \bigcup_{\substack{u \in A^k, m, m' \in M \\ \mu(u)m'm \in \phi(K)}} \pi^{-1}\left(u\mu^{-1}(m)\right) \cap \overline{\pi}^{-1}\left(\overline{\mu}^{-1}(m')\right) \cap \Omega_k\,.$$

Once more, call the left and right hand side X and Y, respectively. Clearly, $X, Y \subseteq \pi^{-1}\left(A^{\geq k}\right) \cap \Omega_k$. Consider some $q \in \pi^{-1}\left(A^{\geq k}\right) \cap \Omega_k$. It suffices to show that $q \in X$ precisely if $q \in Y$.

Since $|\pi(q)| \geq k$, there is a prefix $u \in A^k$ of $\pi(q)$. Lemma 9.8 provides us with $p \in \mathcal{Q}$ satisfying $q = [u]\, p$. According to the motivation of wrw-recognizability right below Theorem 9.4, there are $x, y, z \in A^*$ with $p = [x\overline{y}z]$. Notice that $q = [ux\overline{y}z]$. Since $|M| = k$, there is $y_0 \in A^{\leq k}$ such that $\phi(\overline{y_0}) = \phi(\overline{y})$. Due to $|u| = k \geq |y_0|$ and Corollary 3.8, we conclude $ux\overline{y_0}z \equiv u\overline{y_0}xz$. Combining these facts yields

$$\begin{aligned}
q \in L \quad &\iff \quad \phi(ux\overline{y}z) \in \phi(K) && \text{since } q = [ux\overline{y}z] \\
&\iff \quad \phi(ux\overline{y_0}z) \in \phi(K) && \text{since } \phi(ux\overline{y}z) = \phi(ux\overline{y_0}z) \\
&\iff \quad [ux\overline{y_0}z] \in L && \\
&\iff \quad [u\overline{y_0}xz] \in L && \text{since } ux\overline{y_0}z \equiv u\overline{y_0}xz \\
&\iff \quad \phi(u\overline{y_0}xz) \in \phi(K) && \\
&\iff \quad \phi(u\overline{y}xz) \in \phi(K) && \text{since } \phi(u\overline{y_0}xz) = \phi(u\overline{y}xz)\,.
\end{aligned}$$

Moreover, we have

$$\phi(u\overline{y}xz) = \mu\,(u)\;\overline{\mu}\,(\overline{\pi}(q))\;\mu\left(u^{-1}\pi(q)\right)\,.$$

As we assumed that $q \in \pi^{-1}(A^{\geq k}) \cap \Omega_k$, we obtain

$$q \in X \quad \iff \quad q \in L \quad \iff \quad \mu\,(u)\;\overline{\mu}\,(\overline{\pi}(q))\;\mu\left(u^{-1}\pi(q)\right) \in \phi(K)\,.$$

Finally, utilizing $m = \mu\left(u^{-1}\pi(q)\right)$ and $m' = \overline{\mu}\,(\overline{\pi}(q))$ reveals that the last condition above is equivalent to $q \in Y$. ☐

Lemma 9.11. *Let $k \in \mathbb{N}$ and $L \subseteq \mathcal{Q}$. If L is wrw-recognizable, then the following set is simple:*

$$L \cap \Omega_k \setminus \Omega_{k+1}\,.$$

Proof. Let K, ϕ, M, μ, and $\overline{\mu}$ be as in the proof of Lemma 9.9. We show the claim by establishing the equation

$$L \cap \Omega_k \setminus \Omega_{k+1} = \bigcup_{\substack{u \in A^k, m, m' \in M \\ \mu(u) m' m \in \phi(K)}} \pi^{-1}\left(u\mu^{-1}(m)\right) \cap \overline{\pi}^{-1}\left(\overline{\mu}^{-1}(m')\right) \cap \Omega_k \setminus \Omega_{k+1}\,.$$

Again, call the two sides X and Y, respectively. Clearly, $X, Y \subseteq \Omega_k \setminus \Omega_{k+1}$. Consider some $q \in \Omega_k \setminus \Omega_{k+1}$. It suffices to show that $q \in X$ precisely if $q \in Y$.

Since $q \notin \Omega_{k+1}$, there is $p_0 \in \mathcal{Q}$ with $p_0 \sim_\pi q$, $\mathsf{ow}(p_0) \leq k+1$, and $\mathsf{ow}(p_0) > \mathsf{ow}(q)$. As $\mathsf{ow}(p_0) \leq k$ would contradict $q \in \Omega_k$, we have $\mathsf{ow}(p_0) = k+1$ and hence $\mathsf{ow}(q) \leq k$. Thus, there are $u \in A^k$ and $a \in A$ such that ua is a prefix of $\pi(p_0) = \pi(q)$ and a suffix of $\overline{\pi}(p_0) = \overline{\pi}(q)$. In particular, u is a prefix of $\pi(q)$ and by Lemma 9.8 there is $p \in \mathcal{Q}$ with $q = [u]\,p$. There are $x, y, z \in A^*$ with $p = [x\overline{y}z]$. Notice that $q = [ux\overline{y}z]$, a is a prefix of xz, and ua is a suffix of y. Due to the latter and $\mathsf{ow}(q) \leq k$, a cannot be a prefix of x, i.e., $x = \varepsilon$. Altogether, we obtain

$$\begin{aligned}
q \in X \quad &\iff \quad \phi(u\overline{y}z) \in \phi(K) && \text{since } q = [u\overline{y}z] \\
&\iff \quad q \in Y && \text{since } \phi(u\overline{y}z) = \mu\,(u)\;\overline{\mu}\,(\overline{\pi}(q))\,\mu\left(u^{-1}\pi(q)\right),
\end{aligned}$$

where the last equivalence again uses $m = \mu\left(u^{-1}\pi(q)\right)$ and $m' = \overline{\mu}\,(\overline{\pi}(q))$. ☐

Proposition 9.12. *Every wrw-recognizable subset $L \subseteq \mathcal{Q}$ is simple.*

Proof. Suppose that $\eta^{-1}(L) \cap A^*\overline{A}^*A^*$ is recognizable by a monoid with k elements. Since $\mathcal{Q} = \Omega_0 \supseteq \Omega_1 \supseteq \cdots \supseteq \Omega_k$, we have

$$L = \left(L \cap \pi^{-1}\left(A^{<k}\right) \cap \Omega_k\right) \cup \left(L \cap \pi^{-1}\left(A^{\geq k}\right) \cap \Omega_k\right) \cup \bigcup_{0 \leq \ell < k} \left(L \cap \Omega_\ell \setminus \Omega_{\ell+1}\right) .$$

By Lemmas 9.9, 9.10, and 9.11, the right hand side is a finite union of simple sets and therefore a simple set itself. ☐

We are now prepared to prove the main result of this section.

Proof (of Theorem 9.4). We establish the circular chain of implications "(1)⇒(2)⇒(4)⇒(1)" as well as the equivalence "(1)⇔(3)".

To "(1)⇒(2)" and "(1)⇒(3)". Since L is recognizable, $\eta^{-1}(L)$ is regular and the claims follow.

To "(2)⇒(4)". This is precisely the statement of Proposition 9.12.

To "(4)⇒(1)". For regular $L \subseteq A^*$, the sets $\pi^{-1}(L)$ and $\overline{\pi}^{-1}(L)$ are recognizable. The sets Ω_k with $k \in \mathbb{N}$ are recognizable by Proposition 9.7. Since the class of recognizable subsets of $\mathcal{Q}$ is closed under Boolean combinations, the claim follows.

To "(3)⇒(1)". Let $K = \eta^{-1}(L) \cap \overline{A}^*A^*\overline{A}^*$. Then

$$\delta(K) = \delta\left(\eta^{-1}(L)\right) \cap \delta\left(\overline{A}^*A^*\overline{A}^*\right) = \eta^{-1}\left(\delta(L)\right) \cap A^*\overline{A}^*A^* .$$

Since K is regular, $\delta(K)$ is regular as well and the already established implication "(2)⇒(1)" yields that $\delta(L)$ is recognizable. Finally, this implies that L is recognizable. ☐

10 Thurston-automaticity

Many groups of interest in combinatorial group theory turned out to be Thurston-automatic [CEH+92]. The more general concept of a Thurston-automatic semigroup was introduced in [CRRT01]. In this chapter, we prove that the monoid of queue actions $\mathcal{Q}$ does not fall into this class.

Let Γ be an alphabet and $\diamond \notin \Gamma$. Then consider the new alphabet $\Gamma(2,\diamond) = (\Gamma \cup \{\diamond\})^2 \setminus \{(\diamond,\diamond)\}$. We define the *convolution* $\otimes : \Gamma^* \times \Gamma^* \to \Gamma(2,\diamond)^*$ as follows:

$$\varepsilon \otimes \varepsilon = \varepsilon \qquad av \otimes \varepsilon = (a,\diamond)(v \otimes \varepsilon) \qquad \varepsilon \otimes bw = (\diamond,b)(\varepsilon \otimes w)$$
$$av \otimes bw = (a,b)(v \otimes w)$$

for $a, b \in \Gamma$ and $v, w \in \Gamma^*$. If $R \subseteq \Gamma^* \times \Gamma^*$ let

$$R^{\otimes} = \{\, v \otimes w \mid (v, w) \in R \,\}$$

denote the convolution of R. Note that $R^{\otimes}$ is a language over the alphabet $\Gamma(2, \diamond)$.

Let M be a monoid, Γ an alphabet, $\theta \colon \Gamma^+ \to M$ a semigroup morphism, $L \subseteq \Gamma^+$, and $a \in \Gamma$. Then we define:

$$L_a = \left\{\, (u, v) \in L^2 \;\middle|\; \theta(ua) = \theta(v) \,\right\}^{\otimes} .$$

The triple (Γ, θ, L) is an *automatic structure* for the monoid M if θ maps L bijectively onto M and if the languages L and L_a for all $a \in \Gamma$ are regular.[4] A monoid is *Thurston-automatic* if it has some automatic structure.

Two fundamental results on automatic monoids are the following:

Proposition 10.1. *Let M be a Thurston-automatic monoid.*

1. *If (Γ, θ, L) is an automatic structure for M and $b \in \Gamma$, then the language*

   $$\{\, u \otimes v \mid u, v \in L, \theta(ub) = \theta(vb) \,\}$$

   is regular [CRRT01].
2. *If Γ is a finite set and $\mu \colon \Gamma^* \to M$ a surjective morphism, then M admits an automatic structure $(\Gamma \cup \{\iota\}, \theta, L)$ for some $\iota \notin \Gamma$ with $\theta(a) = \mu(a)$ for all $a \in \Gamma$ and $\theta(\iota) = 1$ [DRR99].*

Using only these basic properties of Thurston-automatic monoids (and a simple counting argument), we can show that $\mathcal{Q}$ does not admit an automatic structure.

Theorem 10.2. *The monoid of queue actions $\mathcal{Q}$ is not Thurston-automatic.*

Proof. Aiming towards a contradiction, assume $\mathcal{Q}$ to be Thurston-automatic. Recall that, by the very definition, $\mathcal{Q}$ is generated by the set $\Sigma = A \cup \overline{A}$ and hence the natural morphism $\eta \colon \Sigma^* \to \mathcal{Q}$ is surjective. Throughout this proof, let $a, b \in A$ be two distinct letters. By Prop. 10.1(2), there exists an automatic structure $(\Sigma \cup \{\iota\}, \theta, L)$ with $\theta(c) = \eta(c)$ for all $c \in \Sigma$ and $\theta(\iota) = \eta(\varepsilon)$. Let $\varphi \colon (\Sigma \cup \{\iota\})^* \to \Sigma^*$ be the morphism with $\varphi(c) = c$ for $c \in \Sigma$ and $\varphi(\iota) = \varepsilon$. Since $\varphi(\iota) = \varepsilon$ and since θ agrees with η on Σ^*, we get $\theta(v) = \theta(\varphi(v)) = \eta(\varphi(v))$ for all $v \in (\Sigma \cup \{\iota\})^*$.

By Prop. 10.1(1), the relation

$$R_0 = \{(u, v) \in L^2 \mid \theta(u\overline{b}) = \theta(v\overline{b})\}$$

is synchronously rational. Since φ is a morphism, also the relation

$$R = \{(\varphi(u), \varphi(v)) \mid u, v \in L, \theta(u\overline{b}) = \theta(v\overline{b})\}$$

[4] This is not the original definition from [CRRT01], but it is equivalent by [CRRT01, Prop. 5.4].

is rational [Ber79]. For $(\varphi(u), \varphi(v)) \in R$, we have $\eta(\varphi(u)\bar{b}) = \theta(u\bar{b}) = \theta(v\bar{b}) = \eta(\varphi(v)\bar{b})$ and therefore $|\varphi(u)| = |\varphi(v)|$. It follows that the relation R is synchronously rational [FS93], i.e., that the language $R^{\otimes}$ is regular.

Let $m, n \in \mathbb{N}$. Since $\theta|_L$ maps L bijectively onto $\mathcal{Q}$, there is a unique word $u_{m,n} \in L$ with $\theta(u_{m,n}) = [\overline{a}^m a^n]$. Then we have $\eta(\varphi(u_{m,n})) = \theta(u_{m,n}) = [\overline{a}^m a^n]$. Since $\overline{a}^m a^n$ is the only element of $[\overline{a}^m a^n]$, this implies $\varphi(u_{m,n}) = \overline{a}^m a^n$.

For $q \in \mathcal{Q}$, $\theta(u_{m,n})[\bar{b}] = q[\bar{b}]$ is equivalent to saying $\pi(q) = a^n$ and $\overline{\pi}(q) = a^m$ (the implication "$\Rightarrow$" is trivial since π and $\overline{\pi}$ are morphisms, the converse one follows from Theorem 5.4). Since $q \in \mathcal{Q}$ is determined by the projections and the overlap width $\mathsf{ow}(q)$, there are precisely $\min(m,n)+1$ many elements $q \in \mathcal{Q}$ with $\theta(u_{m,n})[\bar{b}] = q[\bar{b}]$. Since θ is bijective on L, there are precisely $\min(m,n)+1$ many words $v \in L$ with $(u_{m,n}, v) \in R$. Since also φ is injective on L, we get

$$\begin{aligned}
\min(m,n) + 1 &= |\{\varphi(v) \mid (u_{m,n}, v) \in R_0\}| \\
&= |\{w \mid (\varphi(u_{m,n}), w) \in R\}| \\
&= |\{w \mid (\overline{a}^m a^n, w) \in R\}| \,.
\end{aligned}$$

Let $\mathcal{A}$ be a finite deterministic automaton accepting $R^{\otimes}$. For q a state of $\mathcal{A}$ and $m \in \mathbb{N}$, let $l_q(m)$ denote the number of paths from an initial state to q labeled $\overline{a}^m \otimes w'$ for some $w' \in \{a, \overline{a}\}^m$. Similarly, let $r_q(n)$ denote the number of paths from q to some final state labeled $a^n \otimes w''$ for some $w'' \in \{a, \overline{a}\}^n$. Then, for $m, n \in \mathbb{N}$, we have

$$\min(m,n) + 1 = \sum_{q \in Q} l_q(m) \cdot r_q(n)$$

since the sum equals the number of words $\overline{a}^m a^n \otimes w \in R^{\otimes}$.

Since $\mathbb{N}^Q \times \mathbb{N}^Q$, ordered componentwise, is a well-partial order, there are $m < n$ with $l_q(m) \le l_q(n)$ and $r_q(m) \le r_q(n)$ for all $q \in Q$. Note that

$$\sum_{q \in Q} l_q(m) \cdot r_q(m) = \min(m,m) + 1 < \min(n,n) + 1 = \sum_{q \in Q} l_q(n) \cdot r_q(n) \,.$$

Hence there is $q \in Q$ with $l_q(m) < l_q(n)$ or $r_q(m) < r_q(n)$. Assuming the former, we get

$$\begin{aligned}
m + 1 = \min(m,m) + 1 &= \sum l_q(m) \cdot r_q(m) \\
&< \sum l_q(n) \cdot r_q(m) = \min(n,m) + 1 = m + 1 \,,
\end{aligned}$$

a contradiction. In the latter case, we similarly get

$$\begin{aligned}
m + 1 = \min(m,m) + 1 &= \sum l_q(m) \cdot r_q(m) \\
&< \sum l_q(m) \cdot r_q(n) = \min(m,n) + 1 = m + 1 \,,
\end{aligned}$$

again a contradiction. $\square$

Open question. Recently, the notion of an automatic group has been extended to that of Cayley graph automatic groups [KKM11]. This notion can easily be extended to monoids. It is not clear whether the monoid of queue actions is Cayley graph automatic. A way to disprove this would be to show that the elementary theory of its Cayley graph is undecidable.

Note that $\mathcal{Q}$ is not automatic in the sense of Khoussainov and Nerode [KN95]: This is due to the fact that $\eta(A^*)$ is isomorphic to A^* and an element of $\mathcal{Q}$ is in $\eta(A^*)$ if and only if it cannot be written as $r\overline{a}s$ for $r, s \in \mathcal{Q}$ and $a \in A$. Hence, using the $\overline{a}$ for $a \in A$ as parameters, A^* is interpretable in first order logic in $\mathcal{Q}$. Therefore, since A^* is not automatic in this sense [BG04], neither is $\mathcal{Q}$ [KN95].

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