

## WINNING STRATEGIES FOR PSEUDO-TELEPATHY GAMES USING SINGLE NON-LOCAL BOX

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Using a single NL-box, a winning strategy is given for the impossible colouring pseudo-telepathy game for the set of vectors having Kochen-Specker property in four dimension. A sufficient condition is given regarding the structure of the impossible colouring pseudo-telepathy game for general  $d$ -dimension. A winning strategy for this game is then described with single use of NL-box.

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### 1 Introduction

By performing measurement on an entangled quantum system two separate observer can obtain correlations that are nonlocal, in the sense that no local hidden variable (LHV) model can reproduce it. This was first proved by Bell in 1964 in terms of Bell inequality [1]. Later on Clauser, Horne, Shimony and Holt gave an experimental proposition of Bell's inequality which is known as CHSH inequality [2]. According to CHSH inequality all local hidden variable model must satisfy:

$$|\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle| \leq 2$$

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where  $A_1, A_2$  are observables of a spin-half particle in the possession of Alice and  $B_1, B_2$  are observables of a spin-half particle in the possession of Bob. But local measurement carried out on entangled quantum system can reach the value  $2\sqrt{2}$ . Cirelson showed [3] that this is the maximum value attainable by local measurement on entangled quantum system although the maximum nonlocal value of CHSH inequality can reach is 4.

Popescu and Rohrlich [4] asked a very interesting question: why quantum mechanics is not maximally non-local? Is there any stronger correlation than the quantum mechanical ones that does not allow signalling like quantum correlation? They have introduced a hypothetical non-local box (NL box for short) that does not allow signalling, yet violates CHSH inequality maximally. This NL-box has two input bits  $x$  and  $y$ , and yields two output bits  $a$  and  $b$ . The bits  $x$  and  $a$  are in Alice's hand, while  $y$  and  $b$  are in Bob's hand. The box is such that  $a$  and  $b$  are correlated according to simple relation:

$$x \cdot y = a \oplus b,$$

where  $\oplus$  is addition modulo 2. Afterwards many works have been done to characterize the NL-box in order to yield insights about the non-locality aspects of quantum mechanics [5] - [12].

Quantum pseudo-telepathy game [13] provides an intuitive way to understand quantum non-locality. Quantum pseudo-telepathy game is something which can not be won in the classical world without communication but can be won in the quantum world using entangled state without any use of classical communication. Thus, for an observer (ignorant about any sort of non-locality), the reason for winning of the game by the players would imply some *a priori* 'telepathic' connection between the players. Nevertheless, that sort of connection is impossible. Formally, according to ref. [13], a two-party [14] pseudo-telepathy game is given by a six-tuple  $(X_A, X_B, Y_A, Y_B, P, W)$  where  $X_A$  and  $X_B$  are the input sets of parties Alice and Bob respectively,  $Y_A$  and  $Y_B$  are their respective output sets,  $P (\subseteq X_A \times X_B)$  is the set of all promises, and  $W (\subseteq X_A \times X_B \times Y_A \times Y_B)$  is the winning condition. Thus  $W$  is a relation between inputs and outputs that has to be satisfied by Alice and Bob whenever the promise is fulfilled. Once the respective inputs are supplied to Alice and Bob, they will no longer be allowed to communicate until the game is over. In each round of the game, Alice and Bob are supplied with the inputs  $x \in X_A$  and  $y \in X_B$  respectively. Their task is now to produce outputs  $a \in Y_A$  and  $b \in Y_B$  respectively. They will win the round if either  $(x, y) \notin P$  or  $(x, y, a, b) \in W$ . They will win the game if they go on winning round after round. They will have a winning strategy for the game if they are mathematically certain to win the game as long as they have not exhausted all the classical information as well as quantum entanglement (if there is any) shared at the beginning of the game. Note that some observer of the game (other than Alice and Bob) can only have a statistical evidence towards making the hypothesis that Alice and Bob indeed have a winning strategy for the game, if Alice and Bob go on winning the game round after round. Quantum pseudo-telepathy games are proofs of non-locality. Moreover, they are stronger proofs than usual Bell theorems as well as Bell theorems without inequalities [15].

To understand the features of the NL-box it is necessary to understand its power in various quantum information processing protocols already discovered. There are entangled states (both bipartite as well as multi-partite), the measurement correlations of which can be

simulated by one or more than one NL-box [6, 7, 16]. But there are measurement correlations corresponding to some multi-partite entangled states which can not be simulated by NL-boxes [16]. In this context it would be interesting to know which pseudo-telepathy games, can be won by using NL-box. Recently Broadbent and Méthot [17] showed that some of the pseudo-telepathy games can be won with single use of the NL-box where the quantum strategy requires more than a maximally entangled pair of qubits to succeed. It remained unsolved whether impossible colouring pseudo-telepathy game, constructed by using Kochen-Specker theorem, can be won with a single use of the NL-box. The problem, in general, will be extremely difficult as there are various sets of vectors satisfying Kochen-Specker property for Hilbert space of dimension three or more.

In this paper first we shall present a winning strategy of impossible colouring pseudo-telepathy game for the set of 18 vectors having Kochen-Specker property in four dimension with single use of NL-box. Then we discuss some sufficient condition for the winning strategy of impossible colouring pseudo-telepathy game for general  $d$ -dimension with single use of NL-box. We shall then provide a winning strategy for  $d$ -dimensional impossible colouring pseudo-telepathy game using single NL-box whenever the sufficient condition is satisfied.

In section 2, we shall describe the Kochen-Specker theorem in four dimension that uses eighteen vectors from  $R^4$ , and its corresponding impossible colouring pseudo-telepathy game is described in section 3. A winning strategy for this game is described in section 4 using only one NL-box. A winning strategy for the impossible colouring pseudo-telepathy games in  $d$  dimension, each of which satisfies a suitable sufficient condition, is described in section 5 where it uses a single NL-box. Section 6 draws the conclusion, including a discussion on the magic square game in higher dimensions.

## 2 Kochen-Specker Theorem

There exists an explicit, finite set of vectors in Hilbert space with dimension  $d \geq 3$ , that can not be assigned values  $\{0, 1\}$  such that both of the conditions holds:

1. For every complete set of orthogonal basis vectors, only one vector will get value 1.
2. Value assignment of the vectors will be non-contextual.

We call such set of vectors a set with Kochen-Specker property.

### Example:

The following set of 18 (unnormalized) vectors in  $R^4$  appearing in 9 sets of orthogonal basis has Kochen-Specker property [18]. If on the contrary, one assumes that this set satisfy both the conditions (1) and (2), one gets the following equations.

$$V(0, 0, 0, 1) + V(0, 0, 1, 0) + V(1, 1, 0, 0) + V(1, -1, 0, 0) = 1$$

$$V(0, 0, 0, 1) + V(0, 1, 0, 0) + V(1, 0, 1, 0) + V(1, 0, -1, 0) = 1$$

$$V(1, -1, 1, -1) + V(1, -1, -1, 1) + V(1, 1, 0, 0) + V(0, 0, 1, 1) = 1$$

$$V(1, -1, 1, -1) + V(1, 1, 1, 1) + V(1, 0, -1, 0) + V(0, 1, 0, -1) = 1$$

$$\begin{aligned}
V(0, 0, 1, 0) + V(0, 1, 0, 0) + V(1, 0, 0, 1) + V(1, 0, 0, -1) &= 1 \\
V(1, -1, -1, 1) + V(1, 1, 1, 1) + V(1, 0, 0, -1) + V(0, 1, -1, 0) &= 1 \\
V(1, 1, -1, 1) + V(1, 1, 1, -1) + V(1, -1, 0, 0) + V(0, 0, 1, 1) &= 1 \\
V(1, 1, -1, 1) + V(-1, 1, 1, 1) + V(1, 0, 1, 0) + V(0, 1, 0, -1) &= 1 \\
V(1, 1, 1, -1) + V(-1, 1, 1, 1) + V(1, 0, 0, 1) + V(0, 1, -1, 0) &= 1
\end{aligned}$$

Here  $V(0, 0, 0, 1), \dots, V(0, 1, -1, 0)$  denote the values taken from the set  $\{0, 1\}$  and are assigned to the respective vectors  $(0, 0, 0, 1), \dots, (0, 1, -1, 0)$  (of  $\mathbb{R}^4$ ). If one add these nine equations, the left hand side will be even as every vector has appeared twice and their value can be 1 or 0, while the right hand side is obviously odd. It proves that one can not assign values to all vectors satisfying both the conditions.

### 3 Impossible colouring pseudo-telepathy game in 4-dimension

We now turn this Kochen-Specker theorem in to a pseudo-telepathy game as suggested by Brassard et al. [13]. Consider the nine complete orthogonal bases of real vectors in four dimension, described in the above-mentioned example. Denote them by  $S^1, S^2, \dots, S^9$ , where each  $S^J$  contains the following four pairwise orthogonal vectors  $u_1^J, u_2^J, u_3^J$ , and  $u_4^J$  where,  $u_1^1 = u_1^2 = (0, 0, 0, 1)$ ,  $u_2^1 = u_2^5 = (0, 0, 1, 0)$ , etc. Two players, say, Alice and Bob, are far apart from each other such that Alice is supplied with, at random, any one ( $S^k$ , say) of the nine bases mentioned above, while Bob is supplied with, at random, a vector ( $u_m^l$ , say) from the above-mentioned eighteen vectors. The promise of the game is that  $u_m^l$  must be a member of  $S^k$ . This round of the game will be won by Alice and Bob if the following conditions are satisfied:

Alice will have to assign value (0 or 1) to her four vectors  $u_1^k, u_2^k, u_3^k, u_4^k$  and Bob also will have to assign value (0 or 1) to his single vector  $u_m^k$  in such a way that

1. Exactly Alice's four vectors should receive the value,
2. Alice and Bob have to assign same value to their single common vector  $u_m^k$ ,

with the condition that they will not be allowed to have any classical communication after the game starts and until the game is over. Thus they will win the game if they go on winning it for every round of the game. Interestingly, Brassard et al. [13] presented a quantum winning strategy for a general impossible colouring game in  $d$  dimension using  $\log_2 d$  ebits of shared entanglement between Alice and Bob.

### 4 Wining strategy using a single NL-Box

Now we shall present a strategy to win this game by using a single NL-box. If one tries to satisfy all the above nine equations by assigning non-contextual values to the maximum possible no. of vectors, then one would see that seventeen vectors can be assigned non-contextual values and value assignment for the remaining one vector has to be contextual, *i.e.*, one vector out of eighteen has to take value 1 when it occurs in one basis and 0 when it occurs in another basis [19].

Let us now consider a contextual value assignment to the vector  $(0, 1, -1, 0)$ , which appeared in the above-mentioned nine equations twice – once in the basis  $S^6$  and once in  $S^9$ . We call the following (contextual) value assignment strategy for this vector (together with the remaining seventeen vectors) as  $A0$ : The vector  $(0, 1, -1, 0)$  takes value 1 when it occurs in  $S^6$  and 0 when in  $S^9$ ; and the values assigned to the remaining seventeen vectors are done non-contextually. Similarly we consider another contextual value assignment (call it as  $A1$ ) where the vector  $(0, 1, -1, 0)$  take value 1 when it appears in  $S^9$  and 0 when in  $S^6$ , value assignments for the remaining vectors being non-contextual. Let  $B0$  be the strategy where the eighteen vectors  $u_1^1 = u_1^2 = (0, 0, 0, 1)$ ,  $u_2^1 = u_1^5 = (0, 0, 1, 0)$ ,  $u_3^1 = u_3^3 = (1, 1, 0, 0)$ ,  $u_4^1 = u_3^7 = (1, -1, 0, 0)$ ,  $u_2^2 = u_5^2 = (0, 1, 0, 0)$ ,  $u_3^2 = u_8^3 = (1, 0, 1, 0)$ ,  $u_4^2 = u_3^4 = (1, 0, -1, 0)$ ,  $u_1^3 = u_1^4 = (1, -1, 1, -1)$ ,  $u_2^3 = u_1^6 = (1, -1, -1, 1)$ ,  $u_3^4 = u_7^4 = (0, 0, 1, 1)$ ,  $u_4^4 = u_2^6 = (1, 1, 1, 1)$ ,  $u_4^4 = u_4^8 = (0, 1, 0, -1)$ ,  $u_3^5 = u_3^9 = (1, 0, 0, 1)$ ,  $u_4^5 = u_3^6 = (1, 0, 0, -1)$ ,  $u_4^6 = u_4^9 = (0, 1, -1, 0)$ ,  $u_1^7 = u_1^8 = (1, 1, -1, 1)$ ,  $u_2^7 = u_1^9 = (1, 1, 1, -1)$ ,  $u_8^8 = u_2^9 = (-1, 1, 1, 1)$ , appeared above, are assigned the values 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0 respectively. Similarly, let  $B1$  be the strategy where these eighteen vectors (in the same order as above) are assigned the values 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0 respectively. See the tables below for concise description of  $A0$ ,  $A1$ ,  $B0$  and  $B1$ .

Let each of Alice and Bob adopts two strategies:  $A0$  and  $A1$  for Alice and  $B0$  and  $B1$  for Bob. One can now check that if, in any round of the game, Alice adopts the strategy  $A0$  and Bob adopts the strategy  $B0$ , they will win that round of the game for all the cases except when Alice is supplied with the basis  $S^9$  and Bob is supplied with the vector  $u_4^6 = u_4^9 = (0 \ 1 \ -1 \ 0)$ . Same will hold good if, instead, Alice adopts the strategy  $A1$  while Bob adopts  $B1$ . On the other hand both pairs of strategies  $(A0, B1)$  and  $(A1, B0)$  will give the winning condition of the game when Alice is supplied with the basis  $S^9$  and Bob has given the vector  $u_4^6 = u_4^9 = (0 \ 1 \ -1 \ 0)$ . The strategies  $A0, B0, A1, B1$  are given in the following tabular form:

A0	
set	value
$S^1$	1 0 0 0
$S^2$	1 0 0 0
$S^3$	1 0 0 0
$S^4$	1 0 0 0
$S^5$	0 0 1 0
$S^6$	0 0 0 1
$S^7$	1 0 0 0
$S^8$	1 0 0 0
$S^9$	0 0 1 0

B0	
set	value
$S^1$	1 0 0 0
$S^2$	1 0 0 0
$S^3$	1 0 0 0
$S^4$	1 0 0 0
$S^5$	0 0 1 0
$S^6$	0 0 0 1
$S^7$	1 0 0 0
$S^8$	1 0 0 0
$S^9$	0 0 1 1

A1	
set	value
$S^1$	1 0 0 0
$S^2$	1 0 0 0
$S^3$	1 0 0 0
$S^4$	1 0 0 0
$S^5$	0 0 0 1
$S^6$	0 0 1 0
$S^7$	1 0 0 0
$S^8$	1 0 0 0
$S^9$	0 0 0 1

B1	
set	value
$S^1$	1 0 0 0
$S^2$	1 0 0 0
$S^3$	1 0 0 0
$S^4$	1 0 0 0
$S^5$	0 0 0 1
$S^6$	0 0 1 0
$S^7$	1 0 0 0
$S^8$	1 0 0 0
$S^9$	0 0 0 0

Let us now assume that Alice and Bob share an NL-box. Alice and Bob use this NL-box to choose their strategies among those alternatives. The protocol is as follows: if Alice is supplied with one of the first eight bases, *i.e.*,  $S^1, S^2, \dots, S^8$ , then she will provide 0 as input to the NL-box, otherwise she will choose 1 as input. On the other hand if Bob is given the vector  $(0 \ 1 \ -1 \ 0)$  he will provide 1 as input, otherwise he will choose 0 as input to the NL-box. They will now select their strategies according to the outputs of the NL-box, *i.e.*, if

Alice gets 0 (1) as output, she will use the strategy  $A0$  ( $A1$ ). Similarly if Bob gets 0 (1) as output of the NL-box, then he will assign value to the vector given to him according to the strategy  $B0$  ( $B1$ ).

When Alice is told to assign values to the vectors of one of the bases  $S^1, S^2, \dots, S^8$  and Bob to any vector from that basis, the output of the NL-box will be either 0 and 0 or 1 and 1 to Alice and Bob respectively. Accordingly they will adopt either strategy  $(A0, B0)$  or  $(A1, B1)$ . It is easy to verify from the table that, in each case under this scenario, Alice and Bob will assign same value to the vector given to Bob. When Alice's job is to assign values to the vectors in the basis  $S^9$  and Bob to any vector except  $(0, 1, -1, 0)$ , the strategy will again be either  $(A0, B0)$  or  $(A1, B1)$  and again it will work, as described above. Only when Alice is asked to assign values to the vectors from the basis  $S^9$  and Bob for vector  $(0, 1, -1, 0)$ , both will put the input 1 in the NL-box and get either 0 and 1 or 1 and 0 as their respective outputs. Here the strategy will be either  $(A0, B1)$  or  $(A1, B0)$ . The vector  $(0, 1, -1, 0)$  has same value for both the players. So the above-mentioned method produces a winning strategy for the impossible colouring pseudo-telepathy game.

## 5 Winning strategy for $d$ -dimension using NL-box

Constructions of sets of vectors in general  $d$  dimensions (where  $d \geq 3$ ), having Kochen-Specker property, have been done separately by using geometric method [20] and also by extending a construction in dimension  $d$  to dimension  $d + 1$  [21]. Using each of these constructions, the above-mentioned impossible colouring pseudo-telepathy game can be generalized for any set of vectors having Kochen-Specker property for any dimension  $d$  for  $d \geq 3$ . Brassard et al. [13] have shown that if all the vectors are real, then there always exists a quantum strategy to win this game, where Alice and Bob will have to share a maximally entangled state in  $d \otimes d$  of the form

$$|\phi_{AB}\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle_A \otimes |i\rangle_B$$

The most general form of Kochen-Specker (KS) theorem in  $d$  dimension is as follows: Find out  $r$  number of complete orthogonal bases  $B_i = \{v_{ij}^{\vec{v}} : j = 1, 2, \dots, d \text{ and } v_{ij}^{\vec{v}} \cdot v_{ik}^{\vec{v}} = \delta_{jk}\}$  (for  $i = 1, 2, \dots, r$ ) in  $\mathcal{C}^d$  with  $\bigcup_{i=1}^r B_i = \{v_1^{\vec{v}}, v_2^{\vec{v}}, \dots, v_n^{\vec{v}}\}$  ( $= S$ , say), such that one can never assign  $\{0, 1\}$ -values to each of these  $n$  vectors satisfying the following rules:

- (i) One and only one element of  $B_i$  is assigned the value 1 and the rest are assigned the value 0 (for each  $i$ ),
- (ii) each element of  $S$  should be assigned  $\{0, 1\}$ -values non-contextually.

This means that if  $\mathcal{P}$  is a  $\{0, 1\}$ -value assignment strategy for the elements in  $S$ , then (without loss of generality) we will reach at a contradiction for assignment of  $\{0, 1\}$ -values to the elements of each  $B_i$  for  $i = r - k + 1, r - k + 2, \dots, r$ . Thus, for each  $i \in \{r - k + 1, r - k + 2, \dots, r\}$ , either (a)  $B_i \subseteq \bigcup_{i'=1}^{r-k} B_{i'}$  and all the elements of  $B_i$  have been assigned the value 0, or (b) at least two elements of  $B_i \cap (\bigcup_{i'=1}^{r-k} B_{i'})$  have been assigned the value 1.

As in the case of four dimension (discussed earlier), above-mentioned KS theorem can also be turned into a pseudo-telepathy game. To have a winning strategy for this impossible colouring game, let us assume the following (sufficient) condition, to be satisfied by the above-mentioned choice of  $r$  complete orthogonal bases  $B_1, B_2, \dots, B_r$  in  $\mathcal{C}^d$ , forming the  $n$ -element set  $S$ :

*Condition (1):* There exists a subset  $\mathcal{M}$  (of  $S$ ) of  $m$  vectors such that  $\mathcal{M} \cap B_i$  contains atmost one element for each  $i \in \{r - k + 1, r - k + 2, \dots, r\}$  where, in the corresponding  $\{0, 1\}$ -value assignment strategy  $\mathcal{P}$ , condition (i) of the KS theorem holds good if any element  $\vec{v}$  of  $\mathcal{M}$  is assigned the value  $1 - x$  whenever it is supplied from some  $B_j$  for  $j \in \{r - k + 1, r - k + 2, \dots, r\}$ , although it has already been assigned the value  $x$  through  $\mathcal{P}$  while it occurred in some  $B_{i'}$  for  $i' \in \{1, 2, \dots, r - k\}$ . Moreover, if  $1 - x$  is the value assigned to any element  $\vec{v}$  of  $\mathcal{M} \cap (\bigcup_{i=r-k+1}^r B_i)$  during the above-mentioned value assignment procedure, one can then find another  $\{0, 1\}$ -value assignment procedure  $\mathcal{P}'$  in which the value of this  $\vec{v}$  will be  $1 - x$  (non-contextually) whenever it will occur in some  $B_k$  for  $k \in \{1, 2, \dots, r - k\}$ , and its value will be  $x$  (contextually) whenever it will occur in some  $B_l$  for  $l \in \{r - k + 1, r - k + 2, \dots, r\}$  while all other vectors of  $S$  can be assigned  $\{0, 1\}$ -values non-contextually (satisfying condition (i) of the KS theorem).

Although Condition (1) seems to be too restrictive, the sets of vectors chosen in most of the KS theorems, known so far, satisfy it:

**Example 1:** In the proof of KS theorem [18] with 18 vectors in  $R^4$  and using 9 complete orthogonal bases (described in the example of section 2),  $\mathcal{M} = \{(-1, 1, 1, 1) = u_2^8 = u_2^9\}$ ;  $\mathcal{P}$  is the strategy where the vectors  $u_1^1 = u_1^7 = (0, 0, 0, 1)$ ,  $u_1^2 = u_1^5 = (0, 0, 1, 0)$ ,  $u_3^1 = u_3^3 = (1, 1, 0, 0)$ ,  $u_4^1 = u_3^7 = (1, -1, 0, 0)$ ,  $u_2^2 = u_2^5 = (0, 1, 0, 0)$ ,  $u_3^2 = u_3^8 = (1, 0, 1, 0)$ ,  $u_4^2 = u_3^4 = (1, 0, -1, 0)$ ,  $u_1^3 = u_1^4 = (1, -1, 1, -1)$ ,  $u_2^3 = u_1^6 = (1, -1, -1, 1)$ ,  $u_4^3 = u_4^7 = (0, 0, 1, 1)$ ,  $u_2^4 = u_2^6 = (1, 1, 1, 1)$ ,  $u_4^4, u_4^8 = (0, 1, 0, -1)$ ,  $u_3^5 = u_3^9 = (1, 0, 0, 1)$ ,  $u_4^5 = u_3^6 = (1, 0, 0, -1)$ ,  $u_4^6 = u_4^9 = (0, 1, -1, 0)$ ,  $u_1^7 = u_1^8 = (1, 1, -1, 1)$ ,  $u_2^7 = u_1^9 = (1, 1, 1, -1)$  have been assigned the values 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0 respectively, and the vector  $u_2^8 = u_2^9 = (-1, 1, 1, 1)$  is assigned the value 1 when it occurs in the basis  $S^8$  while it is assigned the value 0 when it occurs in  $S^9$ ;  $\mathcal{P}'$  is the strategy where above-mentioned 17 vectors (in the same order, as written above) have been assigned the values 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0 respectively, and the vector  $u_2^8 = u_2^9 = (-1, 1, 1, 1)$  is assigned the value 0 when it occurs in the basis  $S^8$  while it is assigned the value 1 when it occurs in  $S^9$ .

**Example 2:** In Kernaghan's proof of KS theorem with 20 vectors  $\vec{u}_1 = (1, 0, 0, 0)$ ,  $\vec{u}_2 = (0, 1, 0, 0)$ ,  $\vec{u}_3 = (0, 0, 1, 0)$ ,  $\vec{u}_4 = (0, 0, 0, 1)$ ,  $\vec{u}_5 = (0, 0, 1, 1)$ ,  $\vec{u}_6 = (0, 0, 1, -1)$ ,  $\vec{u}_7 = (0, 1, 0, 1)$ ,  $\vec{u}_8 = (0, 1, 0, -1)$ ,  $\vec{u}_9 = (0, 1, 1, 0)$ ,  $\vec{u}_{10} = (0, 1, -1, 0)$ ,  $\vec{u}_{11} = (1, 1, -1, 1)$ ,  $\vec{u}_{12} = (1, 1, 1, -1)$ ,  $\vec{u}_{13} = (1, -1, 0, 0)$ ,  $\vec{u}_{14} = (1, 1, 1, 1)$ ,  $\vec{u}_{15} = (1, 1, -1, -1)$ ,  $\vec{u}_{16} = (1, 0, 1, 0)$ ,  $\vec{u}_{17} = (1, -1, -1, 1)$ ,  $\vec{u}_{18} = (-1, 1, 1, 1)$ ,  $\vec{u}_{19} = (1, -1, 1, 1)$ ,  $\vec{u}_{20} = (1, 0, 0, -1)$  in  $R^4$  involving 11 complete orthogonal bases  $B_1 = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ ,  $B_2 = \{\vec{u}_1, \vec{u}_2, \vec{u}_5, \vec{u}_6\}$ ,  $B_3 = \{\vec{u}_1, \vec{u}_3, \vec{u}_7, \vec{u}_8\}$ ,  $B_4 = \{\vec{u}_1, \vec{u}_4, \vec{u}_9, \vec{u}_{10}\}$ ,  $B_5 = \{\vec{u}_5, \vec{u}_{11}, \vec{u}_{12}, \vec{u}_{13}\}$ ,  $B_6 = \{\vec{u}_6, \vec{u}_{13}, \vec{u}_{14}, \vec{u}_{15}\}$ ,  $B_7 = \{\vec{u}_7, \vec{u}_{15}, \vec{u}_{16}, \vec{u}_{17}\}$ ,  $B_8 = \{\vec{u}_8, \vec{u}_{11}, \vec{u}_{16}, \vec{u}_{18}\}$ ,  $B_9 = \{\vec{u}_9, \vec{u}_{11}, \vec{u}_{19}, \vec{u}_{20}\}$ ,  $B_{10} = \{\vec{u}_{10}, \vec{u}_{14}, \vec{u}_{17}, \vec{u}_{20}\}$ ,  $B_{11} = \{\vec{u}_{11}, \vec{u}_{12}, \vec{u}_{18}, \vec{u}_{19}\}$  [22], the strategy  $\mathcal{P}$  assigns the values 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0 respectively to the vectors  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_{16}, \vec{u}_{18}, \vec{u}_{19}, \vec{u}_{20}$ , and assigns the value 0 to  $\vec{u}_{17}$  when it occurs in  $B_{10}$ , while 1 when it occurs elsewhere. Moreover, the strategy  $\mathcal{P}'$  assigns the values 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 respectively to the vectors  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_{16}, \vec{u}_{18}, \vec{u}_{19}, \vec{u}_{20}$ , and assigns the value 1 to  $\vec{u}_{17}$  when it occurs in  $B_{10}$ , while 0 when it occurs elsewhere. Thus here  $\mathcal{M} = \{\vec{u}_{17}\}$ .

**Example 3:** In the proof of KS theorem with 37 vectors  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_{37}$  in  $R^3$  involving 26 complete orthogonal bases  $B_1 = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ ,  $B_2 = \{\vec{u}_1, \vec{u}_8, \vec{u}_9\}$ ,  $B_3 = \{\vec{u}_2, \vec{u}_6, \vec{u}_7\}$ ,  $B_4 = \{\vec{u}_2, \vec{u}_{14}, \vec{u}_{17}\}$ ,  $B_5 = \{\vec{u}_2, \vec{u}_{15}, \vec{u}_{16}\}$ ,  $B_6 = \{\vec{u}_3, \vec{u}_4, \vec{u}_5\}$ ,  $B_7 = \{\vec{u}_4, \vec{u}_{12}, \vec{u}_{28}\}$ ,  $B_8 =$

$\{\vec{u}_4, \vec{u}_{11}, \vec{u}_{27}\}$ ,  $B_9 = \{\vec{u}_5, \vec{u}_{10}, \vec{u}_{29}\}$ ,  $B_{10} = \{\vec{u}_5, \vec{u}_{13}, \vec{u}_{26}\}$ ,  $B_{11} = \{\vec{u}_6, \vec{u}_{12}, \vec{u}_{25}\}$ ,  $B_{12} = \{\vec{u}_6, \vec{u}_{13}, \vec{u}_{23}\}$ ,  $B_{13} = \{\vec{u}_7, \vec{u}_{10}, \vec{u}_{24}\}$ ,  $B_{14} = \{\vec{u}_7, \vec{u}_{11}, \vec{u}_{22}\}$ ,  $B_{15} = \{\vec{u}_8, \vec{u}_{11}, \vec{u}_{21}\}$ ,  $B_{16} = \{\vec{u}_8, \vec{u}_{13}, \vec{u}_{20}\}$ ,  $B_{17} = \{\vec{u}_9, \vec{u}_{10}, \vec{u}_{19}\}$ ,  $B_{18} = \{\vec{u}_9, \vec{u}_{12}, \vec{u}_{18}\}$ ,  $B_{19} = \{\vec{u}_{14}, \vec{u}_{27}, \vec{u}_{37}\}$ ,  $B_{20} = \{\vec{u}_{14}, \vec{u}_{29}, \vec{u}_{35}\}$ ,  $B_{21} = \{\vec{u}_{15}, \vec{u}_{26}, \vec{u}_{36}\}$ ,  $B_{22} = \{\vec{u}_{15}, \vec{u}_{28}, \vec{u}_{34}\}$ ,  $B_{23} = \{\vec{u}_{16}, \vec{u}_{19}, \vec{u}_{33}\}$ ,  $B_{24} = \{\vec{u}_{16}, \vec{u}_{21}, \vec{u}_{31}\}$ ,  $B_{25} = \{\vec{u}_{17}, \vec{u}_{18}, \vec{u}_{32}\}$ ,  $B_{26} = \{\vec{u}_{17}, \vec{u}_{20}, \vec{u}_{30}\}$  [Schütte's tautology, described on page 91 of Ref. [23]], the strategy  $\mathcal{P}$  assigns the values 1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0 to the vectors  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_{12}, \vec{u}_{14}, \vec{u}_{15}, \dots, \vec{u}_{37}$  respectively, it assigns the value 0 to the vector  $\vec{u}_{13}$  when it occurs in  $B_{12}$ , while assigns the value 1 for all other occurrence. Moreover, the strategy  $\mathcal{P}'$  assigns the values 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 1, 0, 1 to the vectors  $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_{12}, \vec{u}_{14}, \vec{u}_{15}, \dots, \vec{u}_{37}$  respectively, it assigns the value 1 to the vector  $\vec{u}_{13}$  when it occurs in  $B_{12}$ , while assigns the value 0 for all other occurrence. Thus here  $\mathcal{M} = \{\vec{u}_{13}\}$ .

Alice now chooses one of the two strategies  $A0$  and  $A1$ , while Bob chooses one of two strategies  $B0$  and  $B1$ , described below (in order to have a winning strategy (using single NL box) for the impossible colouring game in  $d$  dimension having the property described by Condition (1)).

$A0$ : When the basis  $B_i$  is supplied, where  $i \in \{1, 2, \dots, r-k\}$ , Alice assigns  $\{0, 1\}$ -values to all of its vectors according to  $\mathcal{P}$ . When the basis  $B_i$  is supplied, where  $i \in \{r-k+1, r-k+1, \dots, r\}$ , Alice assigns the  $\{0, 1\}$ -values (1) to all the vectors of  $(B_i - \mathcal{M})$  according to  $\mathcal{P}$ , (2) to all the vectors of  $B_i \cap (\mathcal{M} - \bigcup_{j=1}^{r-k} B_j)$  according to  $\mathcal{P}$ , and (3) to all the vectors of  $B_i \cap (\mathcal{M} \cap (\bigcup_{j=1}^{r-k} B_j))$  the  $\{0, 1\}$ -values opposite to their respective values (while using  $\mathcal{P}$ ) when they occur in  $B_1, B_2, \dots, B_{r-k}$ .

$A1$ : Same as  $A0$  with  $\mathcal{P}$  replaced by  $\mathcal{P}'$ .

$B0$ : Bob assigns  $\{0, 1\}$ -values to the vectors in  $(S - \mathcal{M})$  according to  $\mathcal{P}$ , while he assigns values to the vectors in  $\mathcal{M}$  according to  $A1$ , assuming that each vector of  $\mathcal{M}$  has been supplied from one of the  $k$  bases  $B_{r-k+1}, B_{r-k+2}, \dots, B_r$ .

$B1$ : Bob assigns  $\{0, 1\}$ -values to the vectors in  $(S - \mathcal{M})$  according to  $\mathcal{P}'$ , while he assigns values to the vectors in  $\mathcal{M}$  according to  $A0$ , assuming that each vector of  $\mathcal{M}$  has been supplied from one of the  $k$  bases  $B_{r-k+1}, B_{r-k+2}, \dots, B_r$ .

The protocol for winning the impossible colouring game will then work as follows:

When Alice is supplied with one of the first  $k$  bases  $B_1, B_2, \dots, B_k$ , she will give 1 as the input to the NL-box, otherwise she will give the input 0. When Bob is supplied with any member of  $\mathcal{M}$ , he will give 1 as the input to the NL-box, otherwise he will give the input 0. They will use their strategies ( $A0$  or  $A1$  by Alice and  $B0$  or  $B1$  by Bob) according to the outputs of the NL-box, as described in the case of the example.

Existence of a classical deterministic winning strategy for the impossible colouring game (*i.e.*, a strategy which does not use entanglement or NL-box or any communication but where Alice can assign  $\{0, 1\}$ -values to the vectors of each supplied orthogonal basis (appeared in the associated Kochen-Specker theorem)  $x$  and Bob can also assign  $\{0, 1\}$ -values to each supplied vector (which is a member of  $x$ ) such that both the conditions in the Kochen-Specker theorem are satisfied) would amount to contradict the Kochen-Specker theorem itself. Hence such a strategy can not exist.

## 6 Conclusion

One should note that in the discussion of the sufficient conditions for having non-local winning strategy for the impossible colouring pseudo-telepathy game in  $d$  dimension, we have not mentioned that vectors have to be real. That condition is necessary for quantum protocol [13] but has no relevance for protocol using NL-box. This difference may be important to understand particular features of quantum entanglement in the context of general non-local theory with the constraint of no signalling.

Another interesting point to be noted in this game is that for some of the KS sets, the classical success probability (with no communication) for impossible colouring pseudo-telepathy game increases with increase in the size of the inputs which is in contrast with most of the other pseudo-telepathy game where classical success probability decreases with the input size. But in this case if the existing quantum protocol is considered, more entanglement is necessary to win the game as the size of the input increases (so classical success probability also increases). Similar thing happens also for pseudo-telepathy game involving magic square of dimension  $n$  ( $n$  being odd). We have checked that magic square game for general odd dimension can easily be reduced to the magic square game in 3-dimension. The classical success probability for the magic square game is given by  $(1 - 1/n^2)$ , which increases as  $n$  increases. Hence for general magic square game 2-e-bit entanglement is sufficient to win if existing quantum protocol is used. It remains a challenge to see whether there exist different winning quantum protocols where amount of entanglement decreases as the classical success probability increases in both the cases.

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